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# Apodisation in Šolc filters: treatment in the complex plane

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**Abstract.** For the theoretical study of Šolc filters, used in particular in solar observatories or coronagraphs, we emphasize the interest of the polarization representation using the complex plane. We show that, with some approximations allowed by the small angle of rotation between two successive filter plates, the traditional but highly computational study of basic Šolc filters is much shorter, and that a new path for the study of less basic Šolc filters is offered by the technique of calculation in the complex plane. This provides a stronger mathematical basis for previous research into tunable or apodising Šolc filters – which is welcome as the relative height of the secondary maxima just near the transmission peaks of basic Šolc filters is 11%.

**Key words.** Polarization, birefringent filters, Šolc filters, apodisation

## 1. Introduction

Birefringent filters, which include those of Lyot and Šolc, consist of polarizers (or analysers) and a fairly large number  $N$  of retardation plates of a birefringent material, cut parallel to its optic axis [1]. They have a spectral transmission curve showing a sequence of very fine peaks [2-5], and this is of great interest for solar observations and coronagraphy [6-9], for telecommunications [10], for sensors [11], etc. – for more details, see the introduction and the references of [1]. Compared to Lyot filters, Šolc filters have the advantage of using only one polarizer before the stack, one analyser at the end and  $N$  identical plates (whence they are simpler and more luminous [5]), but, in their basic forms, they suffer from secondary maxima three times higher than those of Lyot filters just next to the spectral transmission peaks; hence the compelling need of an apodisation [12], *i.e.* a reduction of the secondary maxima – a classical task in image formation and in spectroscopy [13].

The analytical theory of Šolc filters is computationally much heavier than that of Lyot filters. For both basic models of Šolc filters, a rigorous analytical expression of their spectral

transmittance can still be obtained. This becomes much more difficult, if not impossible, for apodising Šolc filters, so most works proceed with numerical studies. A heuristic way of identifying the physical ideas governing these apodising Šolc filters has been presented by Fredga and Högbom [14], and exploited by Leroy [15], but it still involves many computational difficulties. Therefore, in this article, we present a novel approach to the theory of Šolc filters, which lends itself more easily to the study of apodising filters.

In order to go beyond the basic study of polarization problems, three approaches are open [16-18] in the case of non-depolarizing and linear devices: (i) a matrix approach, where polarization is represented by a vector, and the action of an optical component by multiplication by a JONES matrix, which for non-dichroic devices is proportional to a matrix of the  $SU(2)$  unitary group; (ii) a geometrical approach, where a polarization is represented by a point of the Poincaré sphere  $(\Sigma)$ , and the action of a retarder plate by a rotation of  $(\Sigma)$  on itself, or by an element of the rotation group  $SO_3(\mathbb{R})$ ; (iii) an algebraic approach, where a perfect polarization is represented by a point of the complex plane, and the action of a linear non-depolarizing device by a complex homography. Although in practice these three approaches lead to very different calculations, their respective fecundities are fundamentally linked, because of the group morphisms of  $SU(2)$  towards  $SO_3(\mathbb{R})$ , and of  $SO_3(\mathbb{R})$  towards the homography group – the latter resulting in the stereographic projection of  $(\Sigma)$ , the pole of this projection being the point of  $(\Sigma)$  representing the linear polarization parallel to  $Oy$  [17].

As regards the study of Šolc filters, most of the works use the first or second approach (see e.g. [4,19]); it seems that the third approach has hardly been used. This is why, with this third approach, we return to the study of these filters, and in particular to the question of their apodisation and their tuning.

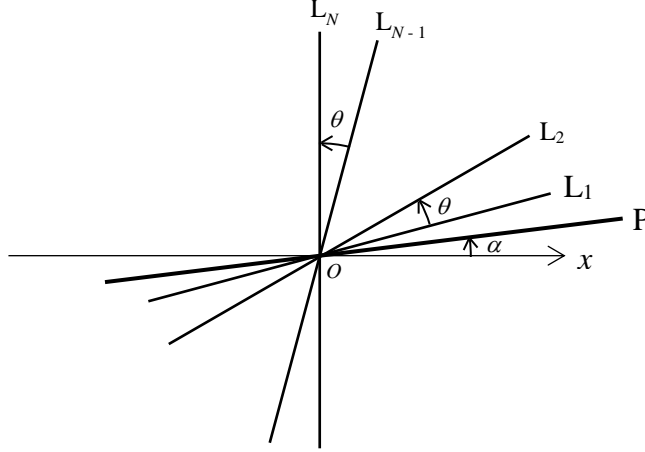
But before tackling the issue, a useful preliminary will introduce the treatment of polarisation in the complex plane by showing how, in this way, we obtain the classical results for the two basic models of Šolc filters – the fan filter and the folded filter [1-4,20]. Both are made up of  $N$  identical retardation plates, normal to  $Oz$ , each producing a retardation (phase shift)  $2\gamma$  of the electric field component  $\mathbf{E}$  (of a monochromatic beam passing through the filter) on its slow axis relative to the component on its fast axis.

For each of these filters, we consider the effect of light propagating through them in the direction of  $Oz$ ; the axes to be considered for all the plates are parallel to the  $xOy$  plane. In order to obtain the expression of the spectral transmittance  $T$  of the filter (which depends on the orientations of the polarizer and of the analyser), according to the third approach we use the correspondence [16-18] between, on the one hand, the polarization of a spectral component (proportional to  $e^{-i\omega t}$ ) of the electric field, which is written  $\underline{\mathbf{E}} = \underline{E}_x \mathbf{u}_x + \underline{E}_y \mathbf{u}_y$  using the direct orthonormal basis  $(\mathcal{B}) := (\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z)$  (the symbol  $:=$  indicating a notation or a definition), and on the other hand a point of affix  $Z := \underline{E}_y / \underline{E}_x$  in the complex plane, extended (*i.e.* compactified by adding infinity, in order to be able to take into account the linear polarisation parallel to  $Oy$ ). The regular arrangement of the successive plates brings back the essence of the problem to the study of a recurrent sequence of complex numbers.

In a first and a second step, we shall revisit, using the complex plane, the study of the fan Šolc filter, then the folded Šolc filter. We use each of these classical cases, for which a rigorous expression of  $T$  is available, as test benches for an approximate calculation allowing us to deal with less classical and more difficult cases. Finally, in a third step, we shall derive an approximate expression of  $T$  for less classical Šolc filters, using this technique of study and approximation in the complex plane – which has apparently never been used before for Šolc filters, and which supports but also refines the heuristic presentation of Fredga and Högbom [14] while avoiding the computational difficulties encountered by Leroy [15].

## 2. Basic fan Šolc filter

In this filter, the azimuth (with respect to  $Ox$ ) of the slow axis  $L_j$  of the retardation plate number  $n$  is  $\theta_n = n\theta$  defined only modulo (see figure 1).



**Figure 1.** Orientations of the slow axes  $L_n$  of the  $N$  plates of the fan filter and of the polarizer axis  $P$ . In the usual configuration the axes of the polarizer and of the analyser  $A$  are parallel to each other ( $\beta = \alpha \text{ mod. } \pi$ ), which explains why  $A$  is not represented here.

### 2.1. Relationship between incident and emergent polarizations for the plate stack

To begin with, we look at the effect of the passage of light through  $L_1$ , of incident polarization characterised by  $Z$  with the basis  $(\mathcal{B})$ . For this purpose, we introduce the direct orthonormal basis  $(\mathcal{B}_1) := (\mathbf{u}'_x, \mathbf{u}'_y, \mathbf{u}'_z)$  such that its first and second vector are respectively directed by the slow and fast axes of  $L_1$ . For light incident on  $L_1$ , the change of basis from  $(\mathcal{B})$  to  $(\mathcal{B}_1)$  changes  $Z$  into  $\frac{Z - \tan \theta}{1 + Z \tan \theta}$ , which must then be multiplied by  $u := e^{-i2\gamma}$ , to account for the passage of the light through  $L_1$ . Let us note that to the first order in  $\theta$ , we have  $e^{-i2\gamma} \frac{Z - \tan \theta}{1 + Z \tan \theta} \cong u \left[ Z - (1 + Z^2) \theta \right]$ .

Then, with a fan filter, the complex number characterising the perfect polarization of the light emerging from plate number  $n$  is therefore – in the direct orthonormal basis  $(\mathcal{B}_n)$  such that its first and second vectors are respectively directed by the slow and fast axes of  $L_n$  – given by the recurrence relation

$$Z_n = u \frac{Z_{n-1} - \tan \theta}{1 + Z_{n-1} \tan \theta}. \quad (1)$$

In practice, the angle between the slow axes of two successive plates is always small, so the first-order approximation in  $\theta$  is correct when  $|Z_{n-1} \theta| \ll 1$ . Under this condition, it gives

$$Z_n \cong u \left[ Z_{n-1} - (1 + Z_{n-1}^2) \theta \right]. \quad (2)$$

By recurrence, it is easy to check that

$$\begin{aligned} Z_n &\cong u^n Z_0 - (u + u^2 + \dots + u^n) (1 + u^{n-1} Z_0^2) \theta \\ &= u^n Z_0 - \frac{u - u^{n+1}}{1 - u} (1 + u^{n-1} Z_0^2) \theta \end{aligned} \quad (3)$$

where  $Z_0$  corresponds, in  $(\mathcal{B})$ , to the perfect polarization of the light incident on  $L_1$ .

The rigorous expression of  $Z_n$  as a function of  $Z_0$  is longer to obtain. It requires to solve a homographic sequence [1]. We can check (see appendix A) that the rigorous expression (A2) is in agreement, to first order in  $\theta$ , with the approximate expression (3) for  $\gamma \in ]0, \pi[$  (where  $u \neq 1$ ). Using this approach, in section 4, we will consider – to first order in  $\theta$  – more difficult practical cases, such as those of apodising Šolc filters, for which there is no rigorous solution. Nevertheless, we must keep in mind a practical problem: the domain of  $\theta$  on which this first order approximation is correct depends on  $\gamma$ , and the values of  $\theta$  corresponding to the usual configuration of the filters (see subsection 3.2) do not allow using the approximate result (3) to calculate  $Z_N$  when  $\gamma$  is too close to the bounds of  $]0, \pi[$ .

## 2.2. Transmittance in the usual configuration

Let us consider a perfect analyser A which transmits without absorption the light whose complex electric field is proportional to  $\underline{a}$  of components  $(\underline{a}_x, \underline{a}_y, 0)$  in  $(\mathcal{B})$ ; the perfect polarization of the latter being characterised by  $Z_A := \underline{a}_y / \underline{a}_x$ . With A, the spectral transmittance  $T$  for a light of perfect polarization characterised by  $Z$  in the orthonormal basis  $(\mathcal{B})$  reads

$$T = \frac{|1 + Z Z_A^*|^2}{(1 + |Z|^2)(1 + |Z_A|^2)} \quad (4)$$

according to a generalized Malus' law in the complex plane [1]. This result is clearly related to the fact that the polarizations (not necessarily linear) associated with  $Z$  and  $Z_A$  are orthogonal if and only if  $Z Z_A^* = -1$  [16-18].

With a basic fan Šolc filter, the axis direction of its analyser A being given by  $\beta = (\underline{u}_x, \underline{n})$ , and when the linear polarization delivered by its polarizer is characterized by  $Z_0 = \tan \alpha$ , the transmittance  $T$  is obtained using the expression (A2) for  $Z$  – with  $n = N$  – and the expression (4) where we express  $Z_A$  using the same basis as for  $Z_N$ , so the basis  $(\mathcal{B}_N)$ ; then  $Z_A = \tan(\beta - N\theta)$ . We have already shown [1] that, with the variable

$$\chi := \arccos(\cos \gamma \cos \theta), \quad (5)$$

this method gives the classical expression [4,20]

$$T = \left[ \frac{\tan \theta}{\tan \chi} \sin(N\chi) \right]^2 \quad (6)$$

in the usual configuration where  $\alpha = \theta/2 = \beta$  and  $N\theta = \frac{\pi}{2} + m\pi$  ( $m \in \mathbb{Z}$ ); in this case

$$Z_0 = \tan \frac{\theta}{2}, \quad \text{and} \quad Z_A = \tan \left( \frac{\theta}{2} - \frac{\pi}{2} - m\pi \right) = -\cot \frac{\theta}{2}.$$

Considering the length of the corresponding calculations [1], it is helpful to examine how they are shortened to first order in  $\theta$ . Thanks to relation (3), for  $\gamma \in ]0, \pi[$  except too close to the bounds, we can write

$$Z_N \cong e^{-i2N\gamma} Z_0 + (A_N + B_N Z_0^2) \theta \quad (7)$$

with

$$A_N = -e^{-i2\gamma} \frac{1 - e^{-i2N\gamma}}{1 - e^{-i2\gamma}} = e^{i2(N-1)\gamma} B_N. \quad (8)$$

For the usual configuration,  $Z_0 \cong \theta/2 \cong -1/Z_A$ , and the term  $B_N Z_0^2 \theta$  is of order 3 in  $\theta$ . To first order, it remains

$$\begin{aligned} Z_N &\cong e^{-i2N\gamma} \frac{\theta}{2} + A_N \theta = \frac{e^{-i2N\gamma} - 2e^{-i2\gamma} + e^{-i2(N+1)\gamma}}{2(1 - e^{-i2\gamma})} \theta = \frac{e^{-i2N\gamma} (e^{i\gamma} + e^{-i\gamma}) - 2e^{-i\gamma}}{2(e^{i\gamma} - e^{-i\gamma})} \theta \\ &= \frac{e^{-i2N\gamma} \cos \gamma - e^{-i\gamma}}{2i \sin \gamma} \theta \end{aligned} \quad (9)$$

Moreover, with  $Z_0$  of order 1 in  $\theta$ , we have  $1 + |Z_N|^2 \cong 1$  for  $\gamma \in ]0, \pi[$ , except too close to the bounds. This exception is here due to the fact that, when  $\gamma$  tends to 0 modulo  $\pi$ , the limit of  $1 + |Z_N|^2$  is notably greater than unity. Other approximation:  $1 + |Z_A|^2 \cong |Z_A|^2$ , because  $Z_A$  is very large (since  $Z_0 = -1/Z_A$  is of first order). With these two last approximations, the denominator of the expression (4) of  $T$  reads

$$(1 + |Z_N|^2)(1 + |Z_A|^2) \cong (\theta/2)^{-2}. \quad (10)$$

Furthermore, and again for  $\gamma \in ]0, \pi[$  except too close to the bounds,

$$1 + Z_N Z_A^* \cong 1 - \frac{e^{-i2N\gamma} \cos \gamma - e^{-i\gamma}}{i \sin \gamma} = \frac{1 - e^{-i2N\gamma}}{i \tan \gamma} = 2e^{-iN\gamma} \frac{\sin(N\gamma)}{\tan \gamma}, \quad (11)$$

then we finally get

$$T \cong \left[ \theta \frac{\sin(N\gamma)}{\tan \gamma} \right]^2. \quad (12)$$

This result is well in agreement with expression (6) in the first order approximation in  $\theta$ , except (as announced above) if  $\gamma$  tends to zero (modulo  $\pi$ ). The above approximations do not allow studying transmission peaks. But the secondary maximum –  $n^\circ j$  (small with respect to  $N$  in absolute value) from the transmission peak for  $\gamma = 0$  – occurs when  $N\gamma \cong \pm \frac{2|j|+1}{2} \pi$ , i.e.

$\gamma \cong \pm (2|j|+1) \theta$  for the usual case where  $m = 0$ . With these values the expression (12) gives

$$T \cong \left\{ \theta / \tan \left[ (2|j|+1) \theta \right] \right\}^2 \cong \left( \frac{1}{2|j|+1} \right)^2 \quad (13)$$

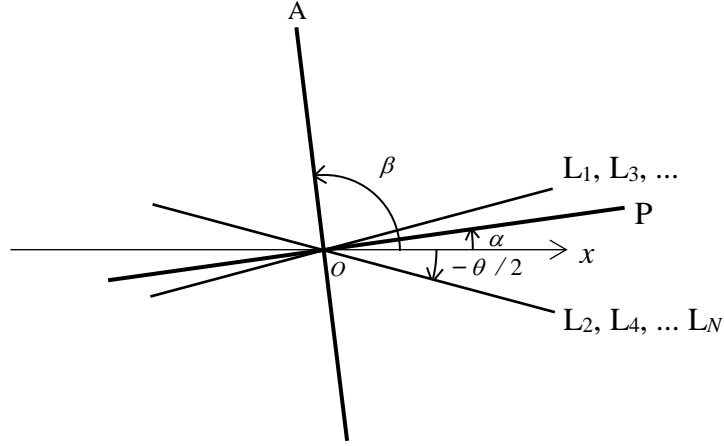
thus  $T \cong 11.1\%$  for  $|j|=1$  and  $m=0$ . Using the rigorous expression (6) Fredga & Högbom [14] give 11.6%, so our approximation appears to be valid to study the quality of apodisation by relying on the value of the two secondary maxima closest to transmission peaks when  $m=0$ .

As the calculations carried out in this way are very shortened, this method seems promising. We will go on with it.

### 3. Study, to first order, of the transmission through a basic folded Šolc filter

In this filter model, the angle (defined only modulo  $\pi$ ) from  $Ox$  towards the slow axis of the retardation plate number  $j$  is  $\theta_n = (-1)^{n-1} \theta/2$ . Orientations of the successive plates are therefore alternating, as they are successively symmetrical with respect to  $Ox$  (figure 2), hence the name “folded Šolc filter”.

Let us limit our study here to the case of such a filter with an even number of plates  $N = 2Q$ . This will allow us to compare our result with another already published [20].



**Figure 2.** Orientations of the slow axes  $L_n$  of the  $N$  plates of the folded filter, of the axes of the polarizer  $P$  and of the analyser  $A$ .

Let us characterize the polarization incident on  $L_1$  by the number  $Z_0$  defined with the components of  $\underline{E}$  in  $(\mathcal{B}_2)$ , and not in  $(\mathcal{B})$  as in section 3. This gives here  $Z_0 = \tan\left(\alpha + \frac{\theta}{2}\right)$ . To find the polarization of the light emerging from  $L_{2Q}$ , let us iterate  $Q$  times the following sequence of calculations:

- transform  $Z$  by the change of basis from  $(\mathcal{B}_2)$  to  $(\mathcal{B}_1)$ , and multiply by  $u$  (in order to know in  $(\mathcal{B}_1)$  the characteristics of the polarization of the light emerging from a plate of odd number), which gives (to first order for the remainder of this article)

$$Z' \cong u \left[ Z - (1 + Z^2) \theta \right]; \quad (14)$$

- transform  $Z'$  by the change of basis from  $(\mathcal{B}_1)$  to  $(\mathcal{B}_2)$ , and multiply by  $u$  (in order to know in  $(\mathcal{B}_2)$  the characteristics of the polarization of the light emerging from the next plate), which gives

$$Z'' \cong u \left[ Z' + (1 + Z'^2) \theta \right] \cong u^2 Z + \left[ (u - u^2) - (u^2 - u^3) Z^2 \right] \theta. \quad (15)$$

In  $(\mathcal{B}_2)$ , the characteristics of the polarization of the light emerging from the last plate of the stack are therefore given by  $Z_Q$  deduced from the following recurrence relation:

$$Z_k \cong u^2 Z_{k-1} + (u - u^2) (1 - u Z_{k-1}^2) \theta \quad (16)$$

(here the subscript  $k$  no longer corresponds to one plate, but to a pair of plates). By recurrence, it is easy to check that

$$Z_k \cong u^{2k} Z_0 + (u - u^2 + u^3 - \dots - u^{2k}) (1 - u^{2k-1} Z_0^2) \theta := u^{2k} Z_0 + (A'_k + B'_k Z_0^2) \theta \quad (17)$$

with

$$A'_k = \frac{u - u^{2k+1}}{1 + u} = -u^{1-2k} B'_k. \quad (18)$$

For the usual configuration  $\alpha = 0$  and  $\beta = \pi/2$ , therefore  $Z_0 \cong \theta/2$  and, still with the basis  $(\mathcal{B}_2)$ ,  $Z_A \cong -2/\theta$ . Then using the result of (17)-(18) we find

$$\begin{aligned} Z_Q &\cong e^{-i2N\gamma} \frac{\theta}{2} + A'_Q \theta = \frac{e^{-i2N\gamma} + 2e^{-i2\gamma} - e^{-i2(N+1)\gamma}}{2(1 + e^{-i2\gamma})} \theta = \frac{e^{-i2N\gamma} (e^{i\gamma} - e^{-i\gamma}) + 2e^{-i\gamma}}{2(e^{i\gamma} + e^{-i\gamma})} \theta \\ &= \frac{e^{-i2N\gamma} i \sin \gamma + e^{-i\gamma}}{2 \cos \gamma} \theta \end{aligned} \quad (19)$$

Moreover,

$$\frac{1}{(1 + |Z_Q|^2)(1 + |Z_A|^2)} \cong \left(\frac{\theta}{2}\right)^2 \quad (20)$$

and

$$1 + Z_Q Z_A^* \cong 1 - \frac{e^{-i2N\gamma} i \sin \gamma + e^{-i\gamma}}{\cos \gamma} = i \frac{1 - e^{-i2N\gamma}}{\cot \gamma} = -2e^{-iN\gamma} \frac{\sin(N\gamma)}{\cot \gamma} \quad (21)$$

from which one finally gets

$$T \cong \left[ \theta \frac{\sin(N\gamma)}{\cot \gamma} \right]^2 = \left\{ \theta \frac{\sin \left[ N \left( \gamma - \frac{\pi}{2} \right) \right]}{\tan \left( \gamma - \frac{\pi}{2} \right)} \right\}^2 \quad (22)$$

for  $N$  even. This result, which corresponds (by changing  $\gamma$  into  $\gamma + \frac{\pi}{2}$ ) to the result (12) obtained for the fan filter, is well in agreement with the expression (24) of [20] to the first-order approximation in  $\theta$ . It necessarily gives  $T \cong 11\%$  for the secondary maxima closest to a principal maximum of  $T$  versus  $\gamma$  on each side of it. As much as for the fan filter, an apodisation for the folded filter will be welcome!

Again, the first-order study of this section is much shorter than the rigorous study – see for example section 3 of [20].

#### 4. Apodising and tuning Šolc filters

In order to obtain apodising filters, Šolc himself proposed [12] to make the angle, between the slow axes of the plates number  $n$  and  $n + 1$  of a fan filter, variable with  $n$ . Here we study the effect of this variation, and find the general idea already perceived by Fredga and Högbom [14], benefitting from the complex plane treatment of polarization. Considering the simple correspondence between the folded and fan filters when used in classical configurations, we will only consider the latter in this last section.

To this end, let us consider  $\varphi_n := \theta_{n+1} - \theta_n$  (with  $\theta_0 := 0$ ), and continue the reasoning of subsection 3.1. For the light incident on  $L_{n+1}$ , the change of basis from  $(\mathcal{B}_n)$  to  $(\mathcal{B}_{n+1})$  – with



$(\mathcal{B}_0) := (\mathcal{B})$  – changes  $Z_n$  into  $Z_n - (1 + Z_n^2)\varphi_n$ , then the passage of the light through  $L_{n+1}$  gives the polarization characterized by

$$Z_{n+1} \cong u \left[ Z_n - (1 + Z_n^2)\varphi_n \right]. \quad (23)$$

Proceeding by recurrence as in section 3, it is easy to check that, always to total order 1 with respect to all the  $\varphi_n$ ,

$$Z_N \cong u^N Z_0 + C_N + D_N Z_0^2. \quad (24)$$

where  $D_N$  is of total order 1, and also  $C_N$  which reads

$$C_N = -u^N \left( \varphi_0 + \varphi_1 u^{-1} + \dots + \varphi_{N-1} u^{-(N-1)} \right) := -u^N \tilde{\varphi}_N \quad (25)$$

where

$$\tilde{\varphi}_N(2\gamma) = \varphi_0 + \varphi_1 e^{i2\gamma} + \dots + \varphi_{N-1} e^{i(N-1)2\gamma} \quad (26)$$

is, as a function of  $2\gamma$ , the discrete (and therefore  $2\pi$ -periodic) Fourier transform of sequence  $\varphi_n$ . Its link with the continuous Fourier transform is well known: samples of the continuous Fourier transform, periodized (its period being the sampling frequency), of an initial function are given by the discrete Fourier transform of the corresponding function periodized – its period being the sampling length [21].

For the usual configuration we have  $Z_0 = -1/Z_A$  real: the corresponding linear polarizations are orthogonal modulo a rotation of angle  $\sum_{n=0}^{N-1} \varphi_n = \theta_N$ , whatever  $\theta_N$  – not necessarily  $\frac{\pi}{2} + m\pi$

(ditto for  $N\theta$  in subsection 3.2); then

$$1 + Z_N Z_A^* \cong -e^{-i2N\gamma} \left( 1 + Z_A \tilde{\varphi}_N - e^{i2N\gamma} \right). \quad (27)$$

Like in subsection 3.2, for  $\gamma \in ]0, \pi[$  except too close to the bounds, and with  $Z_0$  of order 1 in  $\theta_1$ , we will approximate  $(1 + |Z_N|^2)(1 + |Z_A|^2)$  by  $(\theta/2)^{-2}$ , for reasons similar to those exposed just before equation (10). We therefore come to an important relation: for an apodising Šolc filter, in the conditions above,

$$T(2\gamma) \cong \left| 1 - e^{i2N\gamma} + Z_A \tilde{\varphi}_N(2\gamma) \right|^2 / |Z_A|^2 = \left| Z_0 \left( e^{i2N\gamma} - 1 \right) + \tilde{\varphi}_N(2\gamma) \right|^2. \quad (28)$$

In the usual configuration, for the basic case studied in section 3, where  $\varphi_0 = \varphi_1 = \dots = \varphi_{N-1} = \theta = 2\alpha$ , we have

$$\tilde{\varphi}_N(2\gamma) = \theta \frac{\sin(N\gamma)}{\sin\gamma} e^{i(N-1)\gamma} \text{ and } Z_0 \cong \alpha = \theta/2; \quad (29)$$

this simple expression of  $\tilde{\varphi}_N(2\gamma)$  classically appears in the theory of gratings. Then it is easy to check (see appendix B) that the result deduced from the approximate expression (28) is in agreement with (12). Fredga and Högbom [14] show transmission curves for many sequences  $\varphi_n$ .

The approximation  $T(2\gamma) \cong |\tilde{\varphi}_N(2\gamma)|^2$  would be agreeable indeed! It is glimpsed at the end of section 4 and suggested in the beginning of section 5 of Fredga & Högbom [14]. It is correct if the conditions above (hence  $\gamma$  not too close to 0 modulo  $\pi$ ) and  $\left| Z_0 \left( e^{i2N\gamma} - 1 \right) \right| \ll |\tilde{\varphi}_N(2\gamma)|$  are satisfied – often, the latter requires  $\gamma$  not too close to  $\pi/2$  modulo  $\pi$ . Fredga and Högbom missed the existence of such conditions on  $\gamma$ , which allow using the expression  $T(2\gamma) \cong |\tilde{\varphi}_N(2\gamma)|^2$  for the first secondary maxima near the transmission peaks. Hence, this result is valuable to find

sequences  $\varphi_n$  giving interesting results regarding the apodisation – or, heuristically, the tuning [14] – of the transmission curve.

## 5. Conclusion

A new mathematical approach to the question of the apodisation of Šolc filters is offered by the representation of polarization in the complex plane. Taking into account the small angle of rotation between two successive filter plates, not only is the traditional study of fan or folded Šolc filters made much shorter, but a more solid and theoretical basis is given to an approximate result on generalized Šolc filters: their transmittance, for the secondary maxima of transmission (near the peaks) that one wishes to reduce in apodising filters, is quasi-proportional to the square of the modulus of the discrete Fourier transform of sequence  $\varphi_n$ . As already mentioned by Fredga and Högbom [14], this allows the design of multiple types of apodising filters, for example when the sequence of the  $\varphi_n$  is triangular, or Gaussian, etc. Moreover, Fredga and Högbom [14] combine apodisation with tuning of the filter, by changing the  $\varphi_n$  into  $\varphi_n \cos(\omega n)$ : when  $\omega$  varies from 0 to  $\pi/2$ , the principal maxima gets progressively duplicated, and when  $\omega$  varies from  $\pi/2$  to  $\pi$ , the two progressively merge; the filter has been progressively changed from a fan filter into a folded filter.

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## Appendix A

With the variable  $\chi$  defined by equation (5) we have shown [1] that

$$Z_n = \frac{Z_0 [\sin \chi \cos(n\chi) - (\sin \gamma \cos \theta) i \sin(n\chi)] - \sin(n\chi) e^{-i\gamma} \sin \theta}{[\sin \chi \cos(n\chi) + (\sin \gamma \cos \theta) i \sin(n\chi)] + Z_0 \sin(n\chi) e^{i\gamma} \sin \theta}. \quad (\text{A1})$$

We can check that to first order in  $\theta$ , we find the approximate expression (6) for  $\gamma \in ]0, \pi[$ . Indeed, to this order  $\chi \cong \gamma$  because  $\cos \theta \cong 1$  in relation (A1), therefore

$$\sin \chi \cos(n\chi) - (\sin \gamma \cos \theta) i \sin(n\chi) \cong [\cos(n\gamma) - i \sin(n\gamma)] \sin \gamma = e^{-in\gamma} \sin \gamma \quad (\text{A2})$$

and from the relationship (A1) we deduce that

$$\begin{aligned} Z_n &\cong \frac{Z_0 e^{-in\gamma} \sin \gamma - \sin(n\gamma) e^{-i\gamma} \theta}{e^{in\gamma} \sin \gamma + Z_0 \sin(n\gamma) e^{i\gamma} \theta} \\ &\cong \frac{Z_0 e^{-in\gamma} \sin \gamma - \sin(n\gamma) e^{-i\gamma} \theta}{e^{in\gamma} \sin \gamma} \left[ 1 - \frac{Z_0 \sin(n\gamma) e^{i\gamma} \theta}{e^{in\gamma} \sin \gamma} \right] \\ &\cong Z_0 e^{-i2n\gamma} - \frac{\sin(n\gamma)}{e^{in\gamma} \sin \gamma} [e^{-i\gamma} + Z_0^2 e^{-i(2n-1)\gamma}] \theta \\ &= Z_0 e^{-i2n\gamma} - e^{-i2\gamma} \frac{1 - e^{-i2n\gamma}}{1 - e^{-i2\gamma}} [1 + Z_0^2 e^{-i2(n-1)\gamma}] \theta \end{aligned} \quad (\text{A3})$$

as expected from the relationship (3).

## Appendix B

To test the coherence of the approximations given by equations (12) and (28)-(29), we begin with (28). Using the expressions of  $Z_0$  and  $\tilde{\varphi}_N(2\gamma)$  given by equation (29), we find

$$\begin{aligned} T(2\gamma) &= \theta^2 \left| i \sin(N\gamma) e^{iN\gamma} + \frac{\sin(N\gamma)}{\sin \gamma} e^{i(N-1)\gamma} \right|^2 \\ &= \theta^2 \sin^2(N\gamma) \left| i e^{i\gamma} + \frac{1}{\sin \gamma} \right|^2 \\ &= \theta^2 \sin^2(N\gamma) \left| \frac{1 - \sin^2 \gamma + i \sin \gamma \cos \gamma}{\sin \gamma} \right|^2 \\ &= \theta^2 \sin^2(N\gamma) \cos^2 \gamma \left| \frac{\cos \gamma + i \sin \gamma}{\sin \gamma} \right|^2 \\ &= \theta^2 \sin^2(N\gamma) / \tan^2 \gamma \end{aligned} \quad (\text{B1})$$

This result is in agreement with equation (12).