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QUANTUM OPTICS, FROM ONE MODE TO MANY MODES

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QUANTUM OPTICS, FROM ONE MODE TO MANY MODES*

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Abstract

These lectures form a general introduction to the domain of Quantum Optics, with a special emphasis on the notion of field eigenmodes that are necessary to describe the quantum effects in light and define the photons: possibility of changing the basis of field eigenmodes and choosing the most appropriate one, definition of intrinsic properties, independent of the choice of basis. Are successively studied the single mode quantum effects, the quantum correlations existing in bimodal systems, and some interesting features of highly multimode systems such as the optical images.

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1 EIGENMODE BASIS AND QUANTIZATION OF THE FREE RADIATION FIELD

1.1 The analytical signal

1.1.1 Definition

The analytical signal, or complex field, is the generalization of the well known complex representation of an oscillatory phenomenon. Let us consider a real field $\mathbf{E}(\mathbf{r}, t)$. Its Fourier transform is given by:

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \frac{d\omega}{2\pi} \quad (1)$$

Because of the reality of \mathbf{E} , $\tilde{\mathbf{E}}(\omega) = \tilde{\mathbf{E}}^*(-\omega)$. The Fourier decomposition (1) contains therefore redundant information. The analytical signal, or complex field, is defined by:

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \int_0^{\infty} \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \frac{d\omega}{2\pi} \quad (2)$$

It depends only on the spectral components of positive frequency, and therefore has not this problem of redundancy. It allows us to calculate the real field using the relation:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^{(+)}(\mathbf{r}, t) + \mathbf{E}^{(+)*}(\mathbf{r}, t) \quad (3)$$

As we only deal with a free electromagnetic field, there is no static component in it which would pose problems at the boundary $\omega = 0$ of the integral. $\mathbf{E}^{(+)*}(\mathbf{r}, t)$ is often written $\mathbf{E}^{(-)}(\mathbf{r}, t)$.

1.1.2 Evolution equations

As Maxwell equations in vacuum (no charges, no currents) are linear equations with real and time independent coefficients, they are also fulfilled by the corresponding complex fields:

$$\begin{aligned} \nabla \cdot \mathbf{E}^{(+)} &= 0 & \nabla \cdot \mathbf{B}^{(+)} &= 0 \\ \nabla \times \mathbf{E}^{(+)} &= -\frac{\partial}{\partial t} \mathbf{B}^{(+)} & \nabla \times \mathbf{B}^{(+)} &= \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E}^{(+)} \end{aligned} \quad (4)$$

The free fields can be also simply described by the complex vector potential $\mathbf{A}^{(+)}$:

$$\mathbf{E}^{(+)} = -\frac{\partial}{\partial t}\mathbf{A}^{(+)} \quad \mathbf{B}^{(+)} = \nabla \times \mathbf{A}^{(+)} \quad (5)$$

In the *Coulomb gauge*, the Maxwell equations are equivalent to the set:

$$\Delta \mathbf{A}^{(+)} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}^{(+)} = 0 \quad ; \quad \nabla \cdot \mathbf{A}^{(+)} = 0 \quad (6)$$

1.2 Decomposition on the basis of travelling plane waves ("TPW")

1.2.1 Decomposition in Fourier series

The system we are interested in has a finite size, and can be included in a big cubic volume (V) of side L . Inside it, one can define the spatial Fourier components of any complex field $\mathbf{A}^{(+)}(\mathbf{r}, t)$ by:

$$\mathcal{A}_\ell(t) = \frac{1}{L^3} \int_{(V)} d^3r \mathbf{A}^{(+)}(\mathbf{r}, t) \cdot \vec{\epsilon}_\ell e^{-i\mathbf{k}_\ell \cdot \mathbf{r}} \quad (7)$$

which allow us to write any complex field as a linear combination of a discrete set of travelling plane waves (TPW)::

$$\mathbf{A}^{(+)}(\mathbf{r}, t) = \sum_{\ell} \mathcal{A}_\ell(t) \vec{\epsilon}_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}}. \quad (8)$$

The notation \sum_{ℓ} means a sum over a set of integers $\ell = (n_x, n_y, n_z, s)$, where the positive or negative integers n_x, n_y, n_z define the components of the wave vector \mathbf{k}_ℓ :

$$(\mathbf{k}_\ell)_x = n_x \frac{2\pi}{L} \quad ; \quad (\mathbf{k}_\ell)_y = n_y \frac{2\pi}{L} \quad ; \quad (\mathbf{k}_\ell)_z = n_z \frac{2\pi}{L}. \quad (9)$$

and $s = 1, 2$ labels two polarization unit vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ orthogonal to each other and to the wave vector \mathbf{k}_ℓ . These vectors can be complex when one wants to describe circular polarizations.

1.2.2 Time evolution

>From (6), one deduces:

$$\frac{d^2 \mathcal{A}_\ell}{dt^2} = -\omega_\ell^2 \mathcal{A}_\ell \quad (10)$$

where $\omega_\ell = c\|\mathbf{k}_\ell\|$. The transverse plane waves are *harmonic waves*, as is well known. Since $\mathbf{A}^{(+)}$ contains only positive frequencies, the solution $e^{+i\omega_\ell t}$ is excluded, so that:

$$\mathcal{A}_\ell(t) = \underline{\mathcal{A}}_\ell e^{-i\omega_\ell t}. \quad (11)$$

where $\underline{\mathcal{A}}_\ell$ is some complex number. In conclusion, because of the properties of the analytical signal, the evolution equation of $\mathcal{A}_\ell(t)$, instead of (10) is:

$$i \frac{d}{dt} \mathcal{A}_\ell(t) = \omega_\ell \mathcal{A}_\ell(t). \quad (12)$$

It is a *first order differential equation*, rather similar to the Schrödinger equation.

1.2.3 Expression of the physical quantities

One has for the complex electric and magnetic fields:

$$\begin{aligned} \mathbf{E}^{(+)}(\mathbf{r}, t) &= -i \sum_{\ell} \omega_\ell \mathcal{A}_\ell(t) \vec{\varepsilon}_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \\ \mathbf{B}^{(+)}(\mathbf{r}, t) &= i \sum_{\ell} \mathbf{k}_\ell \times \vec{\varepsilon}_\ell \mathcal{A}_\ell(t) e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \end{aligned} \quad (13)$$

The energy H_R of the free field is

$$H_R = \varepsilon_0 \int_{(V)} d^3r (\mathbf{E}^2(\mathbf{r}, t) + c^2 \mathbf{B}^2(\mathbf{r}, t)) \quad (14)$$

Using expressions (13) one can show that :

$$H_R = 2\varepsilon_0 L^3 \sum_{\ell} \omega_\ell^2 |\mathcal{A}_\ell|^2 \quad (15)$$

The energy of the free field appears as a sum of decoupled energies associated with each mode, the "excitation" of each mode being proportional to $|\mathcal{A}_\ell|^2$. The TPW modes therefore "diagonalize" the energy and, as such, are *eigenmodes* of the problem.

1.3 Identification of the canonical conjugate quantities

1.3.1 Usual quantization procedure

Let us consider a classical system depending on some dynamical variables $\{q_i(t), p_i(t)\}$ ($i = 1 \dots n$), the energy of which is given by the hamiltonian $H(q_1, p_1, \dots, q_n, p_n)$. These variables are canonical conjugate variables when the evolution equations of the system are of the Hamilton-Jacobi form:

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i} \quad ; \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i} \quad (16)$$

The quantization procedure associates with these classical, time-dependent, variables $\{q_i(t), p_i(t)\}$ quantum objects which are the hermitian, time independent, operators $\{\hat{q}_i, \hat{p}_i\}$ obeying the following commutation relations:

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{i,j} \quad (17)$$

1.3.2 Case of the free field

Let us introduce the real and imaginary parts of the complex field $\mathcal{A}_\ell(t)$:

$$\mathcal{A}_\ell(t) = \mathcal{A}_{Q\ell}(t) + i\mathcal{A}_{P\ell}(t). \quad (18)$$

They allow us to simply write the energy:

$$H_R = 2\varepsilon_0 L^3 \sum_{\ell} \omega_{\ell}^2 (\mathcal{A}_{Q\ell}^2 + \mathcal{A}_{P\ell}^2) \quad (19)$$

>From (12), one deduces the evolution equations of these real quantities:

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_{P\ell} &= \omega_{\ell}\mathcal{A}_{Q\ell} = \frac{1}{4\varepsilon_0 L^3 \omega_{\ell}} \frac{\partial H_R}{\partial \mathcal{A}_{Q\ell}} \\ \frac{d}{dt}\mathcal{A}_{Q\ell} &= -\omega_{\ell}\mathcal{A}_{P\ell} = -\frac{1}{4\varepsilon_0 L^3 \omega_{\ell}} \frac{\partial H_R}{\partial \mathcal{A}_{P\ell}} \end{aligned} \quad (20)$$

Within a constant multiplicative factor, they are of the hamiltonian form (16). The canonical conjugate quantities can be then chosen as:

$$q_i \leftrightarrow 2\sqrt{\varepsilon_0 L^3 \omega_{\ell}} \mathcal{A}_{Q\ell} \quad ; \quad p_i \leftrightarrow 2\sqrt{\varepsilon_0 L^3 \omega_{\ell}} \mathcal{A}_{P\ell} \quad (21)$$

1.4 Field Quantization on the TPW basis

1.4.1 Commutation relations

We can now associate with these classical quantities time-independent observables $\hat{A}_{q\ell}$ and $\hat{A}_{p\ell}$ obeying the commutation relations:

$$\left[2\sqrt{\varepsilon_0 L^3 \omega_\ell} \hat{A}_{q\ell}, 2\sqrt{\varepsilon_0 L^3 \omega_\ell} \hat{A}_{p\ell'}\right] = i\hbar \delta_{\ell,\ell'} \quad (22)$$

which implies the following relation, which is nothing else than the birth certificate of quantum optics:

$$\left[\hat{A}_{q\ell}, \hat{A}_{p\ell'}\right] = \frac{i\hbar}{4\varepsilon_0 L^3 \omega_\ell} \delta_{\ell,\ell'} \quad (23)$$

Let us introduce the non hermitian operator associated with the complex field amplitude $\hat{A}_\ell = \hat{A}_{q\ell} + i\hat{A}_{p\ell}$. From (23) one deduces:

$$\left[\hat{A}_\ell, \hat{A}_{\ell'}^\dagger\right] = \frac{\hbar}{2\varepsilon_0 L^3 \omega_\ell} \delta_{\ell\ell'} \quad (24)$$

Within a multiplicative factor, it is a relation similar to the familiar commutation relation between an annihilation operator and its hermitian conjugate. If we write:

$$\hat{a}_\ell = \sqrt{\frac{2\varepsilon_0 L^3 \omega_\ell}{\hbar}} \hat{A}_\ell, \quad (25)$$

then one exactly retrieves the commutator algebra of a harmonic oscillator $[\hat{a}_\ell, \hat{a}_{\ell'}^\dagger] = \delta_{\ell\ell'}$.

The complex vector potential operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r})$ can then be written, from (8), as a linear combination of such annihilation operators:

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) = \sum_{\ell} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_\ell L^3}} \hat{a}_{\ell\varepsilon_\ell} e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \quad (26)$$

1.4.2 Hamiltonian of the quantized field

>From (19), we are led to the following expressions:

$$\hat{H}_R = 2\varepsilon_0 L^3 \sum_{\ell} \omega_\ell^2 \left(\hat{A}_{q\ell}^2 + \hat{A}_{p\ell}^2 \right)$$

$$\begin{aligned}
&= \varepsilon_0 L^3 \sum_{\ell} \omega_{\ell}^2 \left(\hat{A}_{\ell} \hat{A}_{\ell}^{\dagger} + \hat{A}_{\ell}^{\dagger} \hat{A}_{\ell} \right) \\
&= \sum_{\ell} \frac{\hbar \omega_{\ell}}{2} \left(\hat{a}_{\ell} \hat{a}_{\ell}^{\dagger} + \hat{a}_{\ell}^{\dagger} \hat{a}_{\ell} \right)
\end{aligned} \tag{27}$$

which is nothing else than *the hamiltonian of an assembly of independent quantum harmonic oscillators*, for which the quantities analogous to position and momentum are the real and imaginary parts of the spatial Fourier component \mathcal{A}_{ℓ} of the complex vector potential $\mathbf{A}^{(+)}(\mathbf{r}, t)$.

Let us note that we could have also chosen to write the energy in terms of the Fourier components of the complex electric field $\mathcal{E}_{\ell} = i\omega_{\ell} \mathcal{A}_{\ell} = \mathcal{E}_{q\ell} + i\mathcal{E}_{p\ell}$, instead of the vector potential \mathcal{A}_{ℓ} . In this alternative approach, completely equivalent to the previous one, the canonical conjugate variables are:

$$q_i \leftrightarrow 2\sqrt{\frac{\varepsilon_0 L^3}{\omega_{\ell}}} \mathcal{E}_{p\ell} \quad ; \quad p_i \leftrightarrow 2\sqrt{\frac{\varepsilon_0 L^3}{\omega_{\ell}}} \mathcal{E}_{q\ell} \tag{28}$$

1.4.3 Stationary states

Stationary states are eigenstates of \hat{H}_R , that one can also write $\sum_{\ell} \hbar \omega_{\ell} (\hat{a}_{\ell}^{\dagger} \hat{a}_{\ell} + 1/2)$. They are then also eigenstates of the "number operator" $\hat{N}_{\ell} = \hat{a}_{\ell}^{\dagger} \hat{a}_{\ell}$. From the commutation relation, one straightforwardly deduces that the spectrum of this operator is the set of natural integers n_{ℓ} . In consequence, there exist states $|n_{\ell}\rangle$ such that:

$$\hat{N}_{\ell} |n_{\ell}\rangle = n_{\ell} |n_{\ell}\rangle \tag{29}$$

A stationary state of the radiation field is a tensor product of such states, $|n_1\rangle \otimes \dots \otimes |n_{\ell}\rangle \dots$, written $|n_1, \dots, n_{\ell}, \dots\rangle$:

$$\hat{H}_R |n_1, \dots, n_{\ell}, \dots\rangle = \sum_{\ell} (n_{\ell} \hbar \omega_{\ell} + 1/2) |n_1, \dots, n_{\ell}, \dots\rangle \tag{30}$$

Due to the additive character of the energies of the radiation field, one can say that the "number state" $|n_1, \dots, n_{\ell}, \dots\rangle$ (or Fock state) contains exactly n_1 independent particles of energy $\hbar \omega_1$, etc, n_{ℓ} independent particles of energy $\hbar \omega_{\ell}$, etc, that are called photons.

1.5 Change of mode basis

So far we have used the basis of travelling plane waves, because they are related to the spatial Fourier transform and allow us to make simple calculations. In contrast, these waves, which have an equal amplitude over all the volume L^3 , are not very physical quantities. We will see now other possible bases.

1.5.1 Unitary transformation of the annihilation and creation operators

Let us perform a *unitary transformation* on the set of travelling plane waves (TPW) creation operators, and define a new set of operators $\{\hat{b}_m^\dagger\}$ by:

$$\hat{b}_m^\dagger = \sum_{\ell} U_m^\ell \hat{a}_\ell^\dagger \quad (31)$$

where U_m^ℓ is an element of a unitary matrix \mathbf{U} ($\mathbf{U}^{-1} = \mathbf{U}^\dagger$). The hermitian conjugate of this quantity is given by:

$$\hat{b}_m = \sum_{\ell} (U_m^\ell)^* \hat{a}_\ell = \sum_{\ell} (U^{-1})_\ell^m \hat{a}_\ell, \quad (32)$$

which enables us to calculate the commutator:

$$[\hat{b}_m, \hat{b}_{m'}^\dagger] = \sum_{\ell, \ell'} (U^{-1})_\ell^m U_{m'}^{\ell'} [\hat{a}_\ell, \hat{a}_{\ell'}^\dagger] = \sum_{\ell} (U^{-1})_\ell^m U_{m'}^\ell = \delta_{m, m'} \quad (33)$$

One sees that the operators $\{\hat{b}_m\}$ form, just like the operators $\{\hat{a}_\ell\}$, a new set of annihilation operators of independent harmonic oscillators.

1.5.2 Field decomposition

Inverting relation (31), one obtains:

$$\hat{a}_\ell = \sum_m U_m^\ell \hat{b}_m \quad (34)$$

This expression of \hat{a}_ℓ , inserted in (26) allows us to obtain an expression of the complex field operator expanded over the new set of annihilation operators $\{\hat{b}_m\}$:

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r}) = \sum_m \hat{b}_m \mathbf{V}_m(\mathbf{r}) \quad (35)$$

where the vectorial function \mathbf{V}_m is given by:

$$\mathbf{V}_m(\mathbf{r}) = \sum_{\ell} \sqrt{\frac{\hbar}{2\varepsilon_0 L^3 \omega_{\ell}}} U_m^{\ell} \vec{\varepsilon}_{\ell} e^{i\mathbf{k}_{\ell} \cdot \mathbf{r}} \quad (36)$$

The set of vectorial functions $\{\mathbf{V}_m\}$ can be used as a basis on which one can decompose any complex field. Each function \mathbf{V}_m describes the *spatial mode* attached to the annihilation operator \hat{b}_m . Note that this basis is in general not orthonormal, because of the mode-dependent factor $\sqrt{\omega_{\ell}}$ in the coefficient in front of the unitary transform matrix element U_m^{ℓ} .

1.5.3 Number states in the new basis

One can define new number states $|m : n_m\rangle$ which are the eigenstates of the new number operators $\hat{N}_m = \hat{b}_m^{\dagger} \hat{b}_m$. These states will describe n_m photons in the new mode associated with the annihilation and creation operators \hat{b}_m and \hat{b}_m^{\dagger} and the spatial mode $\{\mathbf{V}_m\}$. These states will form in turn a new basis for the Hilbert space of the quantized electromagnetic field, on which any quantum state $|\Psi\rangle$ of the electromagnetic field can be expanded.

a- *Invariance of the vacuum*

The ground state of the radiation field, called the vacuum state, is the tensor product of vacuum states in all TPW modes u_{ℓ} , and is noted $|0\rangle$. It contains zero photons in all TPW modes and has the property:

$$\forall \ell \quad \hat{a}_{\ell} |0\rangle = 0 \quad (37)$$

This implies that:

$$\forall m \quad \hat{b}_m^{\dagger} \hat{b}_m |0\rangle = 0 \quad (38)$$

$|0\rangle$ is therefore also the tensor product of vacuum states in the $\{\mathbf{V}_m\}$ modes, corresponding to zero photons in all the modes of the new basis: *the vacuum state is invariant over any change of mode basis.*

b- *Relation between the two kinds of number states*

We can now derive the expression of a number state in the new basis $|m : n_m\rangle$:

$$|m : n_m\rangle = \frac{1}{\sqrt{n_m!}} (\hat{b}_m^\dagger)^{n_m} |0\rangle = \frac{1}{\sqrt{n_m!}} \left(\sum_{\ell} U_m^\ell \hat{a}_\ell^\dagger \right)^{n_m} |0\rangle \quad (39)$$

This expression is particularly simple in the case of the single photon state $|m : 1\rangle$:

$$|m : 1\rangle = \hat{b}_m^\dagger |0\rangle = \sum_{\ell} U_m^\ell \hat{a}_\ell^\dagger |0\rangle = \sum_{\ell} U_m^\ell |\ell : 1\rangle \quad (40)$$

c- Invariance of the total number of photons

The operator $\hat{N}_{tot} = \sum_{\ell} \hat{a}_\ell^\dagger \hat{a}_\ell$ gives the total number of photons in all the modes in the TPW basis. As the transformation for the operators is unitary, one has:

$$\sum_m \hat{b}_m^\dagger \hat{b}_m = \sum_{\ell} \hat{a}_\ell^\dagger \hat{a}_\ell \quad (41)$$

and therefore that *the total number of photons is the same in any mode basis.*

d- Energy

In contrast the hamiltonian H_R , given by (27) is not invariant against unitary transformations. It is transformed into an expression like $\sum_{m,m'} E_{m,m'} \hat{b}_m^\dagger \hat{b}_{m'}$. The new modes are not necessarily decoupled in energy: they are not always eigenmodes of the radiation field hamiltonian.

1.5.4 New basis of eigenmodes

Let us consider a particular case of unitary transformation like in (27), with the following constraint added:

$$U_m^\ell = 0 \quad \text{if} \quad \omega_\ell \neq \omega_m \quad (42)$$

Such a unitary transformation mixes only modes of equal frequencies. One has then:

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r}) = \sum_m \sqrt{\frac{\hbar}{2\varepsilon_0\omega_m}} \hat{b}_m \mathbf{v}_m(\mathbf{r}) \quad (43)$$

an expression now exactly similar to the TPW decomposition (26), where the vectorial functions \mathbf{v}_m , given by:

$$\mathbf{v}_m(\mathbf{r}) = \sum_{\ell} \sqrt{\frac{1}{L^3}} U_m^\ell \vec{\varepsilon}_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \quad (44)$$

now form an orthonormal basis, satisfying the orthonormality and closure relations:

$$\int_{(V)} d^3r \mathbf{v}_m(\mathbf{r}) \cdot \mathbf{v}_{m'}^*(\mathbf{r}) = \delta_{m,m'} \quad (45)$$

$$\sum_m \mathbf{v}_m(\mathbf{r}) \cdot \mathbf{v}_m^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

The number of photons of a given frequency is an invariant with respect to this subset of transformations, which also leave invariant the expression of the hamiltonian. The new modes are therefore also decoupled: they are *new eigenmodes of the radiation field*. The photons defined on this mode basis are therefore independent point-like particles of energy $\hbar\omega_m$, just like the photons that we had defined at the beginning of this chapter from the TPW basis.

Let us quote, as examples of bases of eigenmodes:

1. the standing plane waves (SPW), that we will study in more detail in the next subsection;
2. the harmonic spherical waves, that are used to define the photons that are emitted by excited atoms in a spontaneous emission process;
3. the Laguerre-Gauss or Hermite-Gauss modes, useful to describe beams of light at the paraxial approximation;
4. other bases, as we will see later, are convenient for simplifying some problems, for example the basis containing the "noise mode" in quantum imaging problems (see section (6.3.3)).

1.5.5 Example: standing plane waves (SPW)

Let us now study a very simple case: we only consider light travelling along the Oz axis and of linear polarisation parallel to Ox . By projecting the field on the Ox axis we can therefore forget the vector character of the electromagnetic field. Such a field can be expanded over a basis of travelling wave modes $e^{ik_\ell z}$ with $k_\ell = n_\ell 2\pi/L$. n_ℓ being a positive or negative integer, these waves travel either in the direction of positive z or negative z . One can now

define new annihilation operators \hat{b}_{m+} and \hat{b}_{m-} by:

$$\begin{aligned}\hat{b}_{m+}^\dagger &= \frac{1}{\sqrt{2}}(\hat{a}_{\ell=m}^\dagger + \hat{a}_{\ell=-m}^\dagger) \\ \hat{b}_{m-}^\dagger &= \frac{1}{\sqrt{2}}(\hat{a}_{\ell=m}^\dagger - \hat{a}_{\ell=-m}^\dagger)\end{aligned}\quad (46)$$

These two relations define a unitary transformation from the set $\hat{a}_{\ell=m}^\dagger, \hat{a}_{\ell=-m}^\dagger$ to the set $(\hat{b}_{m+}^\dagger, \hat{b}_{m-}^\dagger)$. It mixes modes of equal frequencies: according to the previous subsection, the operators defined by (46) are creation operators of photons in the eigenmodes $v_{m\pm}(\mathbf{r})$ defined by:

$$\begin{aligned}v_{m+}(\mathbf{r}) &= \frac{1}{\sqrt{2L^3}}(e^{ik_m z} + e^{-ik_m z}) = \sqrt{\frac{2}{L^3}} \cos k_m z \\ v_{m-}(\mathbf{r}) &= \frac{1}{\sqrt{2L^3}}(e^{ik_m z} - e^{-ik_m z}) = i\sqrt{\frac{2}{L^3}} \sin k_m z\end{aligned}\quad (47)$$

These functions correspond to *standing plane wave eigenmodes* (SPW), and the number states $|m\pm : n\rangle$ describe n photons in the standing waves $v_{m\pm}(\mathbf{r})$.

Using relation (39), we can give the expression of the first number states in the standing wave modes in function of the number states $|\ell = m, n_+\rangle \otimes |\ell = -m : n_-\rangle$ in the travelling wave modes (that we shall write $|n_+, n_-\rangle$ to keep simple notations) :

$$\begin{aligned}|m+ : 1\rangle \otimes |m- : 0\rangle &= \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 1\rangle) \\ |m+ : 0\rangle \otimes |m- : 1\rangle &= \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 1\rangle) \\ |m+ : 2\rangle \otimes |m- : 0\rangle &= \frac{1}{2}|2, 0\rangle + \frac{1}{\sqrt{2}}|1, 1\rangle + \frac{1}{2}|0, 2\rangle \\ |m+ : 0\rangle \otimes |m- : 2\rangle &= \frac{1}{2}|2, 0\rangle - \frac{1}{\sqrt{2}}|1, 1\rangle + \frac{1}{2}|0, 2\rangle \\ |m+ : 1\rangle \otimes |m- : 1\rangle &= \frac{1}{\sqrt{2}}(|2, 0\rangle - |0, 2\rangle)\end{aligned}\quad (48)$$

Note that all these quantum states of light are factorized in the basis of standing wave modes, but entangled in the basis of travelling wave modes. This is a proof that *the entanglement of a quantum state is not necessarily an intrinsic property*.

Looking at the coefficients of the decomposition, one also observes that the standing wave photons are in some way rearranged in the two travelling wave modes by a kind of sorting process: according to the first relation, there is an average of 50% photons in each travelling wave in the standing wave single photon state, and respectively 25%, 50% and 25% of the three possible combinations in the two-photon states.

Let us now focus on the last formula. At first sight, the state $|m+ : 1\rangle \otimes |m- : 1\rangle$, like the others, should expand over the $|0, 2\rangle$, $|1, 1\rangle$ and $|2, 0\rangle$ states with the same probabilities as for the $|m+ : 2\rangle \otimes |m- : 0\rangle$. But it is not the case: the state $|1, 1\rangle$ is missing in the decomposition. So that the superposition of two photons travelling in opposite directions gives the same state as one photon in each the two standing waves.

2 OBSERVABLES

In order to describe the properties of the quantum field, one needs to define the physical observables of the system, which are built from the operators \hat{a}_ℓ and \hat{a}_ℓ^\dagger by quantizing the corresponding classical quantities.

2.1 Constants of the motion

2.1.1 Energy

We have already considered it in subsection (1.4.2)

2.1.2 Momentum

One shows that, because of the translational invariance of free space, the classical field momentum, defined by:

$$\mathbf{P}_R = \varepsilon_0 \int_{(V)} d^3r \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \quad (49)$$

is a constant of motion during the free evolution of the classical field.

This quantity can be expressed in terms of the Fourier components of the complex vector potential. One finds

$$\begin{aligned} \mathbf{P}_R &= 2\varepsilon_0 L^3 \sum_{\ell} \omega_{\ell} |\mathcal{A}_{\ell}|^2 \mathbf{k}_{\ell} \\ &= 2\varepsilon_0 L^3 \sum_{\ell} \omega_{\ell} (\mathcal{A}_{q\ell}^2 + \mathcal{A}_{\mathbf{p}\ell}^2) \mathbf{k}_{\ell} \end{aligned} \quad (50)$$

The field momentum appears as a sum of momenta associated with each TPW mode. The momentum of a single mode is parallel to the wave vector \mathbf{k}_{ℓ} (radiation pressure effect) and to $|\mathcal{A}_{\ell}|^2$, i.e. to the energy H_R . More precisely one has in a given mode $H_R = |\hat{P}_R|c$, an expression reminiscent of an ultra relativistic particle of zero rest mass.

The corresponding quantum operator is given by

$$\begin{aligned} \hat{\mathbf{P}}_R &= 2\varepsilon_0 L^3 \sum_{\ell} \omega_{\ell} \left(\hat{\mathbf{A}}_{q\ell}^2 + \hat{\mathbf{A}}_{\mathbf{p}\ell}^2 \right) \mathbf{k}_{\ell} \\ &= 2\varepsilon_0 L^3 \sum_{\ell} \frac{\omega_{\ell}}{2} \left(\hat{A}_{\ell} \hat{A}_{\ell}^{\dagger} + \hat{A}_{\ell}^{\dagger} \hat{A}_{\ell} \right) \mathbf{k}_{\ell} \end{aligned} \quad (51)$$

$$= \sum_{\ell} \frac{\hbar \mathbf{k}_{\ell}}{2} \left(\hat{a}_{\ell} \hat{a}_{\ell}^{\dagger} + \hat{a}_{\ell}^{\dagger} \hat{a}_{\ell} \right) = \sum_{\ell} \hbar \mathbf{k}_{\ell} \hat{a}_{\ell}^{\dagger} \hat{a}_{\ell}$$

the terms arising from the commutator of a_{ℓ} and \hat{a}_{ℓ}^{\dagger} vanish when they are summed over ℓ because for each mode ℓ there is another mode ℓ' such that $\mathbf{k}_{\ell} = -\mathbf{k}_{\ell'}$.

The TPW number states, defined as energy eigenstates, are therefore also momentum eigenstates. Furthermore, one observes also an additive character of the eigenvalues of the momentum operator. As a result photons defined in the TPW mode labelled by ℓ have also a well defined value of the momentum, which is $\hbar \mathbf{k}_{\ell}$.

This property is specific of the TPW basis. For example, on the SPW basis defined in subsection (1.5.5), one finds:

$$\hat{\mathbf{P}} = \sum_m \hbar k_m \mathbf{e}_z (\hat{b}_{+m}^{\dagger} \hat{b}_{-m} + \hat{b}_{-m}^{\dagger} \hat{b}_{+m}) \quad (52)$$

This expression only contains crossed terms and no photon number operator: the standing wave number states $|m_{\pm} : n\rangle$, defined as eigenstates of the energy, are not eigenstates of the momentum operator, which is not a function of the number operators $\hat{b}_{\pm m}^{\dagger} \hat{b}_{\pm m}$. As could be guessed intuitively, *a standing wave photon does not possess a well defined value of the momentum*: when one measures the momentum, there is a probability 0.5 to find $\hbar \mathbf{k}_{\ell}$, and 0.5 to find $-\hbar \mathbf{k}_{\ell}$.

We can therefore state more generally that *the properties of photons are not intrinsic but depend on the basis of eigenmodes on which they are defined*.

2.1.3 Angular momentum

One shows that, because of the rotational invariance of free space, the classical field angular momentum, defined by:

$$\mathbf{J}_R = \varepsilon_0 \int_{(V)} d^3r \mathbf{r} \times (\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)) \quad (53)$$

is a constant of motion. Using the formula of the double vector product and an integration by parts in which one assumes that the fields are zero on the boundary of volume (V) , one finds that:

$$\mathbf{J}_R = \mathbf{L}_R + \mathbf{S}_R \quad (54)$$

with

$$\begin{aligned}\mathbf{L}_R &= \varepsilon_0 \sum_{j=(x,y,z)} \int d^3r E_j(\mathbf{r}, t) (\mathbf{r} \times \nabla) A_j(\mathbf{r}, t) \\ \mathbf{S}_R &= \varepsilon_0 \int d^3r \mathbf{E}(\mathbf{r}, t) \times \mathbf{A}(\mathbf{r}, t)\end{aligned}\quad (55)$$

\mathbf{L}_R , which depends on the origin of coordinates, is named "orbital angular momentum", whereas \mathbf{S}_R , which does not depend on it, is named "intrinsic angular momentum", or "spin angular momentum". These names are only given because of these analogies, but must not be taken too seriously.

a- Intrinsic angular momentum

\mathbf{S}_R can be decomposed on the TPW basis:

$$\mathbf{S}_R = i\varepsilon_0 L^3 \sum_{\mathbf{n}} \omega_{\mathbf{n}} (\mathcal{A}_{\mathbf{n}} \times \mathcal{A}_{\mathbf{n}}^* - \mathcal{A}_{\mathbf{n}}^* \times \mathcal{A}_{\mathbf{n}}). \quad (56)$$

where the transverse vector $\mathcal{A}_{\mathbf{n}}$ can be decomposed on the basis of polarisation vectors $\vec{\varepsilon}_{n,s}$ ($s = 1, 2$). Let us first assume that these vectors describe linear polarizations, and are therefore real. One then gets

$$\mathbf{S}_R = 2i\varepsilon_0 L^3 \sum_{\mathbf{n}} \omega_{\mathbf{n}} (\mathcal{A}_{n,1}^* \mathcal{A}_{n,2} - \mathcal{A}_{n,1} \mathcal{A}_{n,2}^*) \vec{\varepsilon}_{\mathbf{k}}. \quad (57)$$

where $\vec{\varepsilon}_{\mathbf{k}}$ is the unit vector in the direction of the wavevector \mathbf{k} . From this expression one derives the following expression for the intrinsic angular momentum operator on the linear TPW basis:

$$\hat{\mathbf{S}}_R = i \sum_{\mathbf{n}} \hbar (\hat{a}_{n,1}^\dagger \hat{a}_{n,2} - \hat{a}_{n,1} \hat{a}_{n,2}^\dagger) \vec{\varepsilon}_{\mathbf{k}} \quad (58)$$

This expression only contains crossed terms and no photon number operator. As a result, linearly polarized TPW photons have no well defined intrinsic angular momentum, just like SPW photons have no well defined momentum. When one measures the projection of the angular momentum on the propagation axis, one will find \hbar with a probability 0.5, and $-\hbar$ with a probability 0.5.

Let us now use the decomposition of $\mathcal{A}_{\mathbf{n}}$ on the basis of complex unit vectors:

$$\vec{\varepsilon}_{n,\pm} = \frac{1}{\sqrt{2}} (\vec{\varepsilon}_{n,1} \pm \vec{\varepsilon}_{n,2}), \quad (59)$$

which describe right and left handed circular polarizations. One finds:

$$\mathbf{S}_R = 2\varepsilon_0 L^3 \sum_{\mathbf{n}} \omega_{\mathbf{n}} (\mathcal{A}_{n,+}^* \mathcal{A}_{n,+} - \mathcal{A}_{n,-}^* \mathcal{A}_{n,-}) \vec{\varepsilon}_{\mathbf{k}}. \quad (60)$$

for the classical quantity, and

$$\hat{\mathbf{S}}_R = \sum_{\mathbf{n}} \hbar (\hat{a}_{n,+}^\dagger \hat{a}_{n,+} - \hat{a}_{n,-}^\dagger \hat{a}_{n,-}) \vec{\varepsilon}_{\mathbf{k}} \quad (61)$$

for the operator. The expression is now diagonalized, which allows us to say that circularly polarized TPW photons have a well defined intrinsic angular momentum directed along the direction of propagation, the projection of $\hat{\mathbf{S}}_R$ on this axis being equal to $\pm\hbar$. This is reminiscent of the states $|J = 1, m_J = \pm 1\rangle$ of a spin 1 particle, with the restriction that the state $|J = 1, m_J = 0\rangle$ does not exist for such photons.

b- *Orbital angular momentum*

A simple calculation shows that the classical orbital angular momentum of a given TPW is zero whatever its polarization: in the TPW basis the orbital angular momentum expression is complicated and not very useful as it contains only crossed terms. The expression of \mathbf{L}_R is greatly simplified if one makes the *paraxial approximation*, in which one only considers waves that propagate in the vicinity of a given axis, say Oz . This occurs when the field varies in the xOy plane on characteristic distances which are much larger than the wavelength, so that diffraction plays only a minor role, and the wave is "almost plane". The classical complex vector potential can then be written as:

$$\mathbf{A}^{(+)}(\mathbf{r}, t) = \sum_{\ell} \mathcal{A}_{\ell} \vec{\varepsilon}_{\ell} w_{\ell}(x, y, z) e^{ik_{\ell}(z-ct)} \quad (62)$$

where $\vec{\varepsilon}_{\ell}$ is the polarization unit vector and $w_{\ell}(x, y, z)$ a set of "slowly varying envelopes", varying on distances large compared to the wavelength $\lambda = 2\pi c/\omega_{\ell}$, which can be either the set of Hermite-Gauss modes or Laguerre-Gauss modes.

Let us consider the set of Laguerre-Gauss modes TEM_{lm} , which have the following general form in cylindrical coordinates r, ϕ, z :

$$w_{\ell}(x, y, z) = f_{l,m}(r, z) e^{im\phi} \quad (63)$$

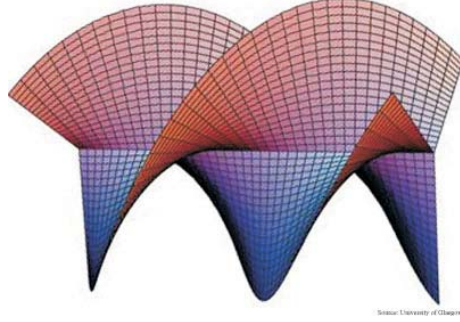


Figure 1: surfaces of equal phase of a Laguerre-Gauss mode of $m = 3$

where m is a relative integer, so that w_ℓ takes the same value for ϕ and $\phi + 2n\pi$. Calculating the integral of (55), one finds:

$$\mathbf{L}_R = 2\varepsilon_0 L^3 \mathbf{e}_z \sum_{\ell} m |\mathcal{A}_\ell|^2 \quad (64)$$

and for the corresponding operator:

$$\hat{\mathbf{L}}_R = \mathbf{e}_z \sum_{\ell} m \hbar \hat{a}_\ell^\dagger \hat{a}_\ell \quad (65)$$

The number states defined on the Laguerre-Gauss basis have therefore, in addition to the spin angular momentum linked to their polarization, a well-defined orbital angular momentum equal to $m\hbar$, related to the spatial shape of the mode in which they are defined, and especially to the phase singularity of the wavefront (helicoidal variation in $e^{im\theta}$ of the mode amplitude, see figure (1)). The photon, which has several properties of a point-like particle, as we will see later, is also able to "feel" the global spatial variation of the mode in which it is defined. It appears in this case more like an elementary excitation of this mode than like a usual elementary particle.

2.2 Field observables

2.2.1 Expressions in the Schrödinger representation

The operators $\hat{\mathbf{A}}^{(+)}(\mathbf{r})$, $\hat{\mathbf{E}}^{(+)}(\mathbf{r})$ et $\hat{\mathbf{B}}^{(+)}(\mathbf{r})$ derived from the classical complex fields are given by:

$$\begin{aligned}\hat{\mathbf{A}}^{(+)}(\mathbf{r}) &= \sum_{\ell} \sqrt{\frac{\hbar}{2\epsilon_0 L^3 \omega_{\ell}}} \hat{a}_{\ell} \vec{\epsilon}_{\ell} e^{i\mathbf{k}_{\ell} \cdot \mathbf{r}} \\ \hat{\mathbf{E}}^{(+)}(\mathbf{r}) &= i \sum_{\ell} \sqrt{\frac{\hbar \omega_{\ell}}{2\epsilon_0 L^3}} \hat{a}_{\ell} \vec{\epsilon}_{\ell} e^{i\mathbf{k}_{\ell} \cdot \mathbf{r}} \\ \hat{\mathbf{B}}^{(+)}(\mathbf{r}) &= i \sum_{\ell} \sqrt{\frac{\hbar}{2\epsilon_0 L^3 \omega_{\ell}}} \hat{a}_{\ell} \mathbf{k}_{\ell} \times \vec{\epsilon}_{\ell} e^{i\mathbf{k}_{\ell} \cdot \mathbf{r}}\end{aligned}\tag{66}$$

The quantity $\sqrt{\hbar \omega_{\ell} / 2\epsilon_0 L^3}$ entering in the expression of the electric field operator will be labelled \mathcal{E}_{ℓ} . It is the "elementary" electric field, yielding a classical electromagnetic energy equal to the energy of a single photon enclosed in volume (V).

All these operators are time independent, the time dependence being carried by the state vector (or density matrix) of the system. The Hermitian operator associated with the classical real field is given by:

$$\hat{\mathbf{E}}(\mathbf{r}) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}) + \hat{\mathbf{E}}^{(-)}(\mathbf{r})\tag{67}$$

$\hat{\mathbf{E}}^{(-)}(\mathbf{r})$ being the hermitian conjugate of $\hat{\mathbf{E}}^{(+)}(\mathbf{r})$, and therefore a linear combination of creation operators \hat{a}_{ℓ}^{\dagger} . Analogous relations hold for the magnetic field and the vector potential.

2.2.2 Expressions in the Heisenberg representation

It can be useful to work in the Heisenberg representation, where the time dependence is now transferred to the operators, the state vector being constant and given by the initial conditions. Let us first look for the expression of the annihilation operator in the Heisenberg representation, $\hat{a}_{H,\ell}(t)$. Its evolution equation is:

$$\begin{aligned}i\hbar \frac{d}{dt} \hat{a}_{H,\ell} &= [\hat{a}_{H,\ell}, \hat{H}_R] = [\hat{a}_{H,\ell}, \hbar \omega_{\ell} \hat{a}_{H,\ell}^{\dagger} \hat{a}_{H,\ell}] \\ &= \hbar \omega_{\ell} \hat{a}_{H,\ell}\end{aligned}\tag{68}$$

One retrieves the temporal evolution of the corresponding classical mode:

$$\hat{a}_{H,\ell}(t) = \hat{a}_\ell e^{-i\omega_\ell t} \quad (69)$$

and for example the electric field operator can be written in the Heisenberg representation as:

$$\hat{\mathbf{E}}_H^{(+)}(\mathbf{r}, t) = i \sum_{\ell} \mathcal{E}_\ell \hat{a}_\ell \vec{\varepsilon}_\ell e^{i(\mathbf{k}_\ell \cdot \mathbf{r} - \omega_\ell t)} \quad (70)$$

which is very close to the expression of the classical complex field decomposed in TPW modes. In the following, we will often forget the index H , when there is no ambiguity concerning the representation in which the quantum field is described.

2.2.3 Single mode field; quadrature operators

Let us call $\hat{\mathbf{E}}_\ell^{(+)}(\mathbf{r}, t)$ the restriction of the complex field operator to a given mode of index ℓ in the Heisenberg representation, and $\hat{E}_\ell^{(+)}$ its projection on the polarization vector $\vec{\varepsilon}_\ell$, which is:

$$\hat{E}_\ell^{(+)}(\mathbf{r}, t) = i\mathcal{E}_\ell \hat{a}_\ell e^{-i\phi_\ell} \quad (71)$$

ϕ_ℓ being the well-known propagation phase $\omega_\ell t - \mathbf{k}_\ell \cdot \mathbf{r}$ of a TPW. One has the following commutation relations:

$$[\hat{E}_\ell^{(+)}(\mathbf{r}, t), \hat{E}_\ell^{(+)}(\mathbf{r}', t')] = 0 \quad ; \quad [\hat{E}_\ell^{(+)}(\mathbf{r}, t), \hat{E}_\ell^{(-)}(\mathbf{r}', t')] = \mathcal{E}_\ell^2 e^{-i(\phi_\ell - \phi'_\ell)} \quad (72)$$

They concern non hermitian operators, which cannot be directly measured. Let us now introduce the following hermitian operators:

$$\hat{E}_{q\ell} = i\mathcal{E}_\ell (a_\ell - a_\ell^\dagger) \quad \hat{E}_{p\ell} = \mathcal{E}_\ell (a_\ell + a_\ell^\dagger) \quad (73)$$

They are the quantum analog of the real and imaginary parts of the complex electric field amplitude in mode ℓ , which are classical canonical conjugate quantities. In the quantum world, they are complementary quantities in Bohr's meaning, analogous to the position and momentum observables of a material particle. They allow us to write the real electric field observable as:

$$\hat{E}_\ell(\mathbf{r}, t) = \hat{E}_{q\ell} \cos \phi_\ell + \hat{E}_{p\ell} \sin \phi_\ell \quad (74)$$

This observable is therefore alternatively equal to $\hat{E}_{q\ell}$ and $\hat{E}_{p\ell}$, when ϕ_ℓ is respectively equal to 0 and $\pi/2 \pmod{2\pi}$. For this reason $\hat{E}_{q\ell}$ and $\hat{E}_{p\ell}$ are called *quadrature operators*, and they obey the following commutation relation:

$$[\hat{E}_{q\ell}, \hat{E}_{p\ell}] = 2i\mathcal{E}_\ell^2 \quad (75)$$

which imply the following Heisenberg inequality, valid for any state of the quantum field:

$$\Delta E_{q\ell} \Delta E_{p\ell} \geq \mathcal{E}_\ell^2 \quad (76)$$

As a result there is no field state which is simultaneously eigenstate of $\hat{E}_{q\ell}$ and $\hat{E}_{p\ell}$, and therefore no state which is eigenstate of the electric field at all points and all times: whatever the quantum state of the radiation field, there always exist points in space and instants in time where the electric field is affected by quantum fluctuations of non zero values. Moreover, if the electric field has vanishingly small fluctuations at a given point and a given time, inequality (76) tells us that it will undergo strong fluctuations a quarter wavelength further, or a quarter period later.

2.3 Photodetection operators

Electric fields in the optical region oscillate 10^{15} times per second, and there is so far no simple instrument which is able to follow such a fast oscillation. The "observable" $\hat{E}_\ell(\mathbf{r}, t)$ is actually not observable in the present state of the technology, and the existing photodetectors have only access to quantities which are averaged over many optical cycles.

2.3.1 Energy detectors

They belong to the category of bolometers, which measure by very sensitive techniques the small temperature increase induced by the absorption of incident light. They measure therefore the energy of the field, and are described more precisely by the observable:

$$\hat{H}_{detected} = \sum_{\ell'} \hbar\omega_{\ell'} \hat{a}_{\ell'}^\dagger \hat{a}_{\ell'} \quad (77)$$

where the sum spans over the modes ℓ' which are actually absorbed in the bolometer.

2.3.2 Photodetectors

Photomultipliers and semiconductor photodiodes directly "transform photons into electrons". They are based on a photo-ionization process induced by the incident light and creating free electrons (or free electron-hole pairs), which are then measured as a photocurrent i_{ph} . The full quantum theory of such a detector was first made by Glauber. Using a perturbation theory at the lowest order, he showed that, if $|\Psi\rangle$ is the quantum state of the field incident on the detector, the mean photocurrent is given by:

$$i_{ph} = \int_{S_D} d^2r \sum_{\ell} R_{\ell} \langle \Psi | \hat{E}_{\ell}^{(-)}(\mathbf{r}, t) \hat{E}_{\ell}^{(+)}(\mathbf{r}, t) | \Psi \rangle \quad (78)$$

where the integral spans over the photodetector surface S_D and R_{ℓ} is the sensitivity factor of the detector to the mode ℓ . This expression gives the mean value of the photocurrent when there are only a few photons which create well separated "clicks" (photon counting regime) as well as when the number of incident photons is large and the clicks overlap, so that the photocurrent varies continuously. It gives only the mean value of the photocurrent, but no information about the statistics of clicks in the first case, nor the value of the photocurrent quantum fluctuations in the second case.

If the detector has a small area S centered on \mathbf{r}_D , is photosensitive only within a small range of frequencies around a value ω_0 , is formed of many absorbers and contains no traps for the created free charges, so that all the incident photons are finally absorbed and create a current (unity quantum efficiency), which is actually the case for some photodiodes in the market, one can define a *photocurrent operator* \hat{i}_{ph} given by:

$$\hat{i}_{ph} = \frac{2q\varepsilon_0 c S}{\hbar\omega_0} \hat{E}^{(-)}(\mathbf{r}_D, t) \hat{E}^{(+)}(\mathbf{r}_D, t) \quad (79)$$

which allows us to determine not only the mean value of the photocurrent, but also its higher statistical moments. It is the quantum analog of the quantity $|E^{(+)}(\mathbf{r}_D, t)|^2$ which is measured in classical optics.

2.3.3 Double photodetection

In order to evaluate the presence of correlations in the radiation field, experiments in quantum optics often rely on *coincidence measurements* of photo-counts, or *intensity correlation measurements* given by the product

of photocurrents, measured by two photodetectors placed at points \mathbf{r}_D and \mathbf{r}'_D . Glauber has shown that the mean coincidence rate, or correlation, $w(\mathbf{r}_D, t; \mathbf{r}'_D, t')$ is given by:

$$w(\mathbf{r}_D, t; \mathbf{r}'_D, t') = \langle \Psi | \hat{E}^{(-)}(\mathbf{r}_D, t) \hat{E}^{(-)}(\mathbf{r}'_D, t') \hat{E}^{(+)}(\mathbf{r}_D, t) \hat{E}^{(+)}(\mathbf{r}'_D, t') | \Psi \rangle \quad (80)$$

Note that the product of field operators in (80) contains products of two annihilation operators. Its mean value is therefore zero in a single photon state: a single photon cannot be detected at two different points, a property which gives to the photon the status of a point-like particle.

2.4 Input-output relations for the field operators

2.4.1 General remarks

The general scheme of an optical experiment is the following: one has input light beams coming from sources such as lasers, which are processed and mixed by an optical set-up, containing linear and nonlinear elements, which produces output beams, on which one can put photodetectors which extract the relevant information. Let us label i the different measured modes at the output. The overall effect of the optical system is that the electric field operator of the i^{th} output mode $\hat{E}_{i,out}^{(+)}$ is a function of all the input field operators and of their hermitian conjugates $\hat{E}_{1,in}^{(+)}, \hat{E}_{1,in}^{(-)}, \hat{E}_{2,in}^{(+)}, \hat{E}_{2,out}^{(-)}, \dots$. Such an *input-output operatorial relation* can take many different forms according to the optical set-up which is used, but not any form, because $\hat{E}_{i,out}^{(+)}$ must be indeed an electric field operator, and therefore obey the commutation relation $[\hat{E}_{i,out}^{(+)}, \hat{E}_{j,out}^{(-)}] = i\mathcal{E}_\ell^2 \delta_{i,j}$: the input-output relations must be *canonical* i.e. preserve the commutation relations of free field operators.

2.4.2 Free propagation

The simplest case is when the field propagates freely from one point \mathbf{r}_{in} to another one \mathbf{r}_{out} . In the basis of TPW modes, the input-output relation between the field operators at these two points is then:

$$\hat{E}_{i,out}^{(+)} = e^{i\phi} \hat{E}_{i,in}^{(+)} \quad (81)$$

where the propagation phase is equation $\phi = \omega_\ell(t_1 - t_2) - \mathbf{k}_\ell \cdot (\mathbf{r}_1 - \mathbf{r}_2)$. This relation obviously preserves the commutation relation.

2.4.3 Beamsplitter

It is the simplest optical device allowing us to mix two modes 1 and 2 of the field of identical frequencies, polarization and transverse spatial variations. If one transposes the classical beamsplitter relations to the corresponding complex field operators, one gets:

$$\begin{aligned}\hat{E}_{1,out}^{(+)} &= t\hat{E}_{1,in}^{(+)} + r\hat{E}_{2,in}^{(+)} \\ \hat{E}_{2,out}^{(+)} &= -r\hat{E}_{1,in}^{(+)} + t\hat{E}_{2,in}^{(+)}\end{aligned}\quad (82)$$

where r and t are real reflection and transmission amplitude coefficients fulfilling $r^2 + t^2 = 1$. The minus sign in front of r in the second equation is necessary to have energy conservation between the two input beams and the two output beams. It is easy to check that relations (82) preserve the commutation relations. As ω_ℓ , and therefore \mathcal{E}_ℓ , is the same for all the modes, one has the same relation for the annihilation operators:

$$\begin{aligned}\hat{a}_{1,out} &= t\hat{a}_{1,in} + r\hat{a}_{2,in} \\ \hat{a}_{2,out} &= -r\hat{a}_{1,in} + t\hat{a}_{2,in}\end{aligned}\quad (83)$$

It is easy to check that this relation preserves the commutation relations.

2.4.4 Two-wave parametric mixing

Let us now consider a nonlinear crystal having a non-negligible $\chi^{(2)}$ coefficient pumped by an intense laser at frequency ω_0 . It will couple by parametric interaction signal and idler modes, labelled 1 and 2, of frequencies ω_1 and ω_2 such that $\omega_1 + \omega_2 = \omega_0$. If one neglects pump depletion, the propagation equations in the nonlinear crystal lead to a simple expression connecting the output signal and idler classical fields to the input ones, which can be extended to the corresponding quantum operators. The expression is particularly simple when one considers the case of perfect phase matching and of a purely imaginary complex pump field:

$$\begin{aligned}\hat{E}_{1,out}^{(+)} &= \hat{E}_{1,in}^{(+)} \cosh S + \hat{E}_{2,in}^{(-)} \beta \sinh S \\ \hat{E}_{2,out}^{(+)} &= \hat{E}_{1,in}^{(-)} \frac{\sinh S}{\beta} + \hat{E}_{2,in}^{(+)} \cosh S\end{aligned}\quad (84)$$

where S is a parameter proportional to $\chi^{(2)}$, the pump amplitude and the crystal length, and $\beta = \sqrt{n_2\omega_1/n_1\omega_2}$ (n_1, n_2 : optical index of the crystal

for the signal and idler modes). Note that in this case $\hat{E}_{1,out}^{(+)}$ is coupled to the hermitian conjugate operator $\hat{E}_{2,in}^{(-)}$. It is simple to check first that this transformation does not conserve the total energy of the signal and idler modes: there is some amount of positive or negative energy transfert from the pump mode to these modes, and secondly that these transformations conserve the commutation relations (having in mind that in a dielectric medium the quantity \mathcal{E}_ℓ is equal to $\sqrt{\hbar\omega_\ell/2n_\ell\varepsilon_0L^3}$, n_ℓ being the optical index of the medium for the mode ℓ). Relations (84) have a simpler expression when they are expressed in terms of the corresponding annihilation operators:

$$\begin{aligned}\hat{a}_{1,out} &= \hat{a}_{1,in} \cosh S + \hat{a}_{2,in}^\dagger \sinh S \\ \hat{a}_{2,out} &= \hat{a}_{1,in}^\dagger \sinh S + \hat{a}_{2,in} \cosh S\end{aligned}\tag{85}$$

2.4.5 Symplectic transformations

Expressions (81), (83) and (85) describe transformations which belong to an important group, called the *symplectic group*, which is made of all the input-output relations which are linear in the annihilation and creation operators and conserve the commutation relations. These transformations correspond to a Hamiltonian evolution in which the hamiltonian is *bilinear with respect to the annihilation and creation operators of the different modes*. It can be shown that any symplectic transformation can be written as a succession of elementary transformations of the three classes given in subsections (2.4.2), (2.4.3) and (2.4.4).

The sub-group consisting of transformations which, in addition to the commutators, conserve the energy, like (81) and (83), is called the group of passive symplectic transformations. Passive symplectic transformations can always be seen as a more or less complicated succession of free propagation and beamsplitter transformations.

2.5 Balanced homodyne detection

Balanced homodyne detection is a technique which is often used in quantum optics, because it allows one to measure the quadrature operators, and also to have access to all the properties of a quantum state of light through the technique of "quantum tomography".

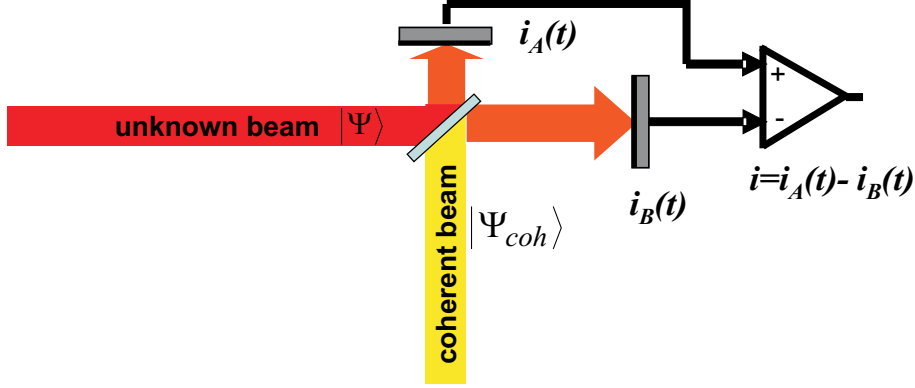


Figure 2: Balance homodyne detection set-up

2.5.1 Description

The set-up is given in figure (2): the field to measure, in quantum state $|\Psi\rangle$ of the input mode $\ell = 1$, is mixed with a single mode local oscillator, supposed to be in a coherent state $|Ae^{i\phi}\rangle$ of a given input mode $\ell = 2$ (see section (4.2.2)), on a beamsplitter of energy transmission and reflection coefficients 50%. The two output beams, labelled (1) and (2), are detected by two perfect photodetectors, and the two photocurrents are subtracted. The resulting photocurrent operator, expressed in terms of photon numbers, \hat{N}_- , is:

$$\begin{aligned}\hat{N}_- &= \hat{a}_{1,out}^\dagger \hat{a}_{1,out} - \hat{a}_{2,out}^\dagger \hat{a}_{2,out} \\ &= \hat{a}_{1,in}^\dagger \hat{a}_{2,in} + \hat{a}_{2,in}^\dagger \hat{a}_{1,in}\end{aligned}\quad (86)$$

using equations (83) with $r = t = 1/\sqrt{2}$. In this expression, $\hat{a}_{1,out}$, $\hat{a}_{2,out}$, $\hat{a}_{1,in}$ and $\hat{a}_{2,in}$ are the annihilation operators of respectively the two output and the two input modes on the beamsplitter. Note that this expression is valid only when the input modes (1) and (2) have exactly the same spatial and temporal dependence, in phase and amplitude, on the mixing beamsplitter, in which case they are said to be exactly "mode-matched".

2.5.2 Mean value and variance

The mean value of the detected signal \hat{N}_- in state $|\Psi\rangle \otimes |Ae^{i\phi}\rangle$ is

$$\langle \hat{N}_- \rangle = A \langle \Psi | \hat{a}_{1,in}^\dagger e^{i\phi} + \hat{a}_{1,in} e^{-i\phi} | \Psi \rangle \quad (87)$$

It gives a quantity proportional to the mean value of the operator $\hat{x}_\phi = \hat{a}_{1,in}^\dagger e^{i\phi} + \hat{a}_{1,in} e^{-i\phi}$, which is proportional to the quadrature operators $\hat{E}_{q\ell}$ and $\hat{E}_{p\ell}$ when the local oscillator phase (that one can tune by scanning the optical length of beam 2) is respectively 0 and $\pi/2$. These two hermitian operators are therefore not only observables in the mathematical meaning, but correspond to quantities the mean value of which can actually be measured. As for the variances, one easily shows that:

$$\Delta^2 N_- = A^2 \Delta^2 x_\phi + \langle \Psi | \hat{a}_{1,in}^\dagger \hat{a}_{1,in} | \Psi \rangle \quad (88)$$

If the mean photon number in the mode to measure is much smaller than the photon number A^2 in the local oscillator mode, then

$$\Delta^2 N_- \approx A^2 \Delta^2 x_\phi \quad (89)$$

A noise measurement in a homodyne detection set-up gives also, within this approximation, the quantum fluctuations of the field quadratures, amplified by the factor A^2 , which can be very large. These fluctuations are therefore easily measurable, even if the mode to measure is in the vacuum state or contain only a few photons.

2.5.3 Quantum tomography

If one records in a computer memory the instantaneous fluctuations of the difference photocurrent N_- , and therefore of the observable \hat{x}_ϕ , on a sufficiently long time interval, one can determine the statistical distribution of this fluctuating quantity and experimentally evaluate the quantity $P_\phi(x)$, probability of measuring a given value x for the quadrature \hat{x}_ϕ . Such an operation can be repeated for many successive values of the local oscillator phase ϕ spanning the range $(0, \pi)$. The information contained in the series of functions $P_\phi(x)$, $\phi \in [0, \pi]$ is enough to completely determine the exact quantum state of the incident light. For example, the following expression, called "inverse Radon transform", which can be numerically evaluated, gives the Wigner function $W(q, p)$ of the incident quantum state (see section (3.1.2)):

$$W(q, p) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} |k| dk \int_0^\pi d\phi P_\phi(x') e^{i2\pi k(\cos \phi + p \sin \phi - x')} \quad (90)$$

This technique of reconstructing the quantum state from its different "sections" measured by balanced homodyne detection, is called "quantum tomography".

3 CHARACTERIZATION OF THE QUANTUM STATES OF LIGHT

To simplify the expressions, we will assume that the quantum state of the radiation field evolves in a space with a finite number N of modes, which can be very large.

3.1 General descriptions

3.1.1 State vector

As the number states $|n_1, \dots, n_N\rangle$ form a basis of the Hilbert space of the quantum radiation states vectors, the most general state vector of light $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} c_{n_1, \dots, n_N} |n_1, \dots, n_N\rangle \quad (91)$$

constrained by the normalization condition $\langle\Psi|\Psi\rangle = 1$. The number of complex numbers c_{n_1, \dots, n_N} , and therefore of different possible quantum states, is gigantic: each number n_ℓ spans over an infinite number of values, and there is a big number of such sums. Let us recall that a classical radiation field is fully characterized when one knows the series of N complex numbers $\mathcal{A}_1, \dots, \mathcal{A}_N$. This means that the diversity of physical situations in quantum optics is prodigious. They have just started to be studied, and situations which are farther and farther from the classical radiation field are regularly found. There is no doubt that much more remains to be found.

Let us calculate the mean value of the complex electric field in the state (91). It is given by:

$$\langle\Psi|\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)|\Psi\rangle = i \sum_{\ell=1}^N \mathcal{E}_\ell \alpha_\ell e^{-i(\omega_\ell t - \mathbf{k}_\ell \cdot \mathbf{r})} \quad (92)$$

where the complex quantity α_ℓ is given by:

$$\alpha_\ell = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} c_{n_1, \dots, n_{\ell-1}, \dots, n_N}^* c_{n_1, \dots, n_\ell, \dots, n_N} \quad (93)$$

Expression (92) is the same as the classical expressions (11,13): the diversity of quantum states cannot be seen on the mean values of the fields. It is only on the variances of the different observables, on the correlation between measurements, and more generally on the higher moments of the quantum probability distribution, that non-classical effects occur. This is what will be studied in the following.

3.1.2 Density matrix

In many physical situations, either one does not completely master the initial conditions, so that the system has some amount of classical fluctuations, or one observes only a part of a big system. In both cases, the physical system is said to be in a *mixed state* instead of a *pure state*. To describe it, one must use the density matrix ρ , which is a hermitian positive operator acting on the Hilbert state of state vectors, and submitted to the constraint $Tr\rho = 1$. The diversity of such density matrices is even more colossal than the one of state vectors.

The parameter which characterizes the amount of "classical noise" in the system is the *purity* $P = tr\rho^2$, a quantity which is always between 0 and 1: when $P = 1$, the state is pure and can be described by a vector like (91). It contains only quantum fluctuations. When $P < 1$ the state is mixed and has also some amount of classical fluctuations. The minimum value of P can be shown to be equal to the inverse of the dimension of the Hilbert space in which the state lives (1/2 for a qubit for example). P can therefore be zero in a Hilbert space of infinite dimensions.

3.1.3 Wigner function

a-Definitions

In quantum optics, the density matrix contains an infinite number of elements and is difficult to manipulate. One often uses instead a quasi-probability distribution function, which is a real function in phase space, depending on $2N$ real variables, and which also completely characterizes the system. There are different ones, with different interesting properties. We will focus here on the Wigner function, because it is the one which is the most adapted to the description of the properties of quadrature operators, and therefore of the homodyne measurements described in subsection (2.5).

Let us first introduce the *displacement operator*, also called Weyl operator.

It is a unitary operator that depends on N complex variables $\lambda_1, \dots, \lambda_N$ and is given by:

$$\hat{D}(\lambda_1, \dots, \lambda_N) = e^{\sum_{\ell} (\lambda_{\ell} \hat{a}_{\ell}^{\dagger} - \lambda_{\ell}^* \hat{a}_{\ell})} \quad (94)$$

If one uses the real and imaginary parts of λ_{ℓ} , $\lambda_{\ell} = u_{\ell} + iv_{\ell}$, this operator can be also written in terms of the $2N$ real variables (u_{ℓ}, v_{ℓ}) :

$$\hat{D}(u_1, v_1, \dots, u_N, v_N) = e^{-i \sum_{\ell} (u_{\ell} \hat{q}_{\ell} + v_{\ell} \hat{p}_{\ell})} \quad (95)$$

where the hermitian operators \hat{q}_{ℓ} and \hat{p}_{ℓ} , which are given by:

$$\hat{q}_{\ell} = i(\hat{a}_{\ell} - \hat{a}_{\ell}^{\dagger}) \quad ; \quad \hat{p}_{\ell} = \hat{a}_{\ell} + \hat{a}_{\ell}^{\dagger} \quad (96)$$

are the dimensionless versions of the field quadrature operators introduced in subsection (2.2.3), that we will also call quadrature operators. They are conjugate quantum operators which obey the following commutation relations and Heisenberg inequality¹:

$$[\hat{q}_{\ell}, \hat{p}_{\ell}] = 2i \quad ; \quad \Delta q_{\ell} \Delta p_{\ell} \geq 1 \quad (97)$$

\hat{D} is the operator which "displaces" the annihilation and quadrature operators:

$$\hat{D} \hat{a}_{\ell} \hat{D}^{\dagger} = \hat{a}_{\ell} + \lambda_{\ell} \quad ; \quad \hat{D} \hat{p}_{\ell} \hat{D}^{\dagger} = \hat{p}_{\ell} + u_{\ell} \quad ; \quad \hat{D} \hat{q}_{\ell} \hat{D}^{\dagger} = \hat{q}_{\ell} + v_{\ell} \quad (98)$$

The Wigner function is nothing else than *the Fourier transform of the mean value of the displacement operator*. It can be written either as a function of N complex variables $\bar{\alpha} \equiv (\alpha_1 \dots \alpha_N)$ or of $2N$ real variables $\bar{R} \equiv (q_1, p_1 \dots q_N, p_N)$ as:

$$W(\bar{\alpha}) = \frac{1}{\pi^{2N}} \int d^2 \lambda_1 d^2 \lambda_N \text{Tr} \left[\rho \hat{D}(\lambda_1 \dots \lambda_N) \right] e^{\sum_{\ell} (\lambda_{\ell} \alpha_{\ell}^* - \lambda_{\ell}^* \alpha_{\ell})} \quad (99)$$

$$W(\bar{R}) = \frac{1}{(2\pi)^{2N}} \int du_1 dv_1 \dots du_N dv_N \text{Tr} \left[\rho \hat{D}(u_1, v_1 \dots u_N, v_N) \right] e^{i \sum_{\ell} (u_{\ell} q_{\ell} + v_{\ell} p_{\ell})} \quad (100)$$

Let us mention that the definitions (99) can also be used by replacing the

¹ \hat{q}_{ℓ} and \hat{p}_{ℓ} differ formally from the usual operators position and momentum of a particle, \hat{x} and \hat{p} , by a factor $\sqrt{2/\hbar}$. Other scaling factors can be found in the literature.

density matrix ρ by any hermitian operator \hat{A} . One defines in such a way the Wigner function $W_{\hat{A}}(\bar{\alpha})$, or $W_{\hat{A}}(\bar{R})$, associated with the observable \hat{A} . One finds easily that:

$$W_{\hat{1}_\ell}(\bar{R}) = \left(\frac{1}{4\pi}\right)^N; \quad W_{\hat{q}_\ell}(\bar{R}) = \left(\frac{1}{4\pi}\right)^N q_\ell; \quad W_{\hat{p}_\ell}(\bar{R}) = \left(\frac{1}{4\pi}\right)^N p_\ell \quad (101)$$

where $\hat{1}_\ell$ is the identity operator in mode ℓ .

b- *Properties*

Most of them are the consequences of an important theorem allowing us to write the trace of the product of two observables in terms of the overlap of their Wigner functions:

$$Tr \hat{A} \hat{B} = (4\pi)^N \int d\bar{R} W_{\hat{A}}(\bar{R}) W_{\hat{B}}(\bar{R}) \quad (102)$$

One deduces that:

- 1) If one takes $\hat{A} = \rho$, $\hat{B} = \hat{1}_\ell$, one finds that the Wigner function of a quantum state is a normalized function:

$$\int d\bar{\alpha} W(\bar{\alpha}) = \int d\bar{R} W(\bar{R}) = 1 \quad (103)$$

- 2) If one takes $\hat{A} = \rho_1$, $\hat{B} = \rho_2$, one finds an expression for the vector product $Tr \rho_1 \rho_2$ between the two states (equal to $|\langle \psi_1 | \psi_2 \rangle|^2$ in the case of two pure states):

$$Tr \rho_1 \rho_2 = (4\pi)^N \int d\bar{R} W_1(\bar{R}) W_2(\bar{R}) \quad (104)$$

This relation has an important consequence: when two pure states are orthogonal, then $\langle \psi_1 | \psi_2 \rangle = 0$, and therefore the overlap integral between the two Wigner functions is zero. This can occur only when some part of these functions is negative. A Wigner function may therefore take either positive or negative values.

- 3) If one takes $\hat{A} = \rho$, $\hat{B} = \rho$, one finds an expression of the purity of the state in function of its Wigner function:

$$P = Tr \rho^2 = (4\pi)^N \int d\bar{R} (W(\bar{R}))^2 \quad (105)$$

- 4) If one takes $\hat{A} = \rho$, $\hat{B} = \hat{q}_\ell$ or $\hat{B} = \hat{p}_\ell$, one finds that the mean value of the quadrature operators is the integral of the corresponding classical quantity weighted by the Wigner function of the state:

$$\langle \hat{q}_\ell \rangle = \int d\bar{R} q_\ell W(\bar{R}) \quad \langle \hat{p}_\ell \rangle = \int d\bar{R} p_\ell W(\bar{R}) \quad (106)$$

The same kind of relation holds when one wants to calculate the mean value of \hat{a}_ℓ in terms of an integral of the complex number α_ℓ weighted by $W(\bar{\alpha})$.

More generally, one can show that the mean value of any symmetrized product of n_1 operators \hat{q}_ℓ and n_2 operators \hat{p}_ℓ , that we will call $S((\hat{q}_\ell)^{n_1}(\hat{p}_\ell)^{n_2})$ ², can be written as an integral over $d\bar{R}$ of the same function of the corresponding classical quantities $q_\ell^{n_1} p_\ell^{n_2}$ weighted by the Wigner function $W(\bar{R})$, and that the mean value of the symmetrized operator $S(\hat{a}_\ell^{n_1}(\hat{a}_\ell^\dagger)^{n_2})$ can be written as an integral over $d\bar{\alpha}$ of the same function of the corresponding classical quantities $\alpha_\ell^{n_1} \alpha_\ell^{*n_2}$ weighted by the Wigner function $W(\bar{\alpha})$. This property induces us to interpret the Wigner function $W(\bar{R})$ as a probability density for the quantities (q_ℓ, p_ℓ) . But it cannot be a classical probability density function, because the variables q_ℓ and p_ℓ are complementary in Bohr's meaning. The corresponding operators do not commute, implying the Heisenberg inequality (76), and there is no quantum state in which these two quantities are perfectly defined simultaneously. The Wigner function has necessarily a finite extension in phase space³. In addition, the Wigner function is not a well-behaved probability function because, as we have seen, it may take negative values. For this reason, it is called a *quasi-probability density function*.

- 5) The relation (106) applied to the operator \hat{q}_1 , for example, can be rewritten as:

$$\langle \hat{q}_1 \rangle = \int dq q P_{q_1}(q) \quad (107)$$

²The symmetrized operator $S((\hat{q}_\ell)^{n_1}(\hat{p}_\ell)^{n_2})$ is obtained by summing all the possible terms equal to the product of n_1 operators \hat{q}_ℓ and n_2 operators \hat{p}_ℓ in any order and dividing the result by the number of terms. Other quasi-probability functions allow us to calculate products of operators in other kinds of order. Let us note that, with the help of the commutation relations, one can transform any operator into a sum of symmetrized terms.

³There is an analogous limitation when one wants to describe a phenomenon simultaneously in the time and in the frequency domain, which are also complementary quantities.

where $P_{q_1}(q)$ is the following quantity:

$$P_{q_1}(q) = \int dp_1 dq_2 dp_2 \dots dp_N dq_N W(q, p_1, q_1, p_2, \dots, q_N, p_N) \quad (108)$$

This quantity, which can be shown to be positive, appears in relation (107) as the true probability to find the value q when one measures the operator \hat{q}_1 . It is actually equal to $|\psi(q)|^2$ in the case where the state is described by a wavefunction $\psi(q)$. Equation (108) shows that it is obtained by integrating the Wigner function over all the variables except the one we are interested in.

- 6) Let us consider a symplectic input-output relation for the fields (see subsection (2.4.5)). This means that the output operators depend linearly on the input operators:

$$\hat{a}_\ell^{out} = \sum_{\ell'} \left(A_{\ell}^{\ell'} \hat{a}_{\ell'}^{in} + B_{\ell}^{\ell'} \hat{a}_{\ell'}^{in\dagger} \right) \quad (109)$$

and that the transformations (109) preserve the commutation relations. One can show that the Wigner function describing the output state of the system can be simply expressed in terms of the Wigner function describing its input state:

$$W^{out}(\bar{\alpha}) = W^{in} \left(\alpha_1 = \sum_{\ell'} \left(A_1^{\ell'} \alpha_{\ell'} + B_1^{\ell'} \alpha_{\ell'}^* \right) \dots \alpha_N = \sum_{\ell'} \left(A_N^{\ell'} \alpha_{\ell'} + B_N^{\ell'} \alpha_{\ell'}^* \right) \right) \quad (110)$$

In other words, in order to obtain the output Wigner function, one takes the input Wigner function and replaces its variables by new values equal to the initial ones transformed by the classical transformation of the field variables. In some way the Wigner distribution behaves as a classical density distribution in a classical input-output relation. But this simple behaviour is only valid for the symplectic transformations.

3.2 The ground state of the radiation field

It is the state containing zero photons in all modes, $|\ell = 1 : 0\rangle \otimes \dots |\ell : 0\rangle \otimes \dots$, noted $|0\rangle$. Usually called the vacuum, it should be called the obscurity,

because it describes the state of light in which all the light sources have been switched off. It is easy to show that it has the following properties:

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r}, t)|0\rangle = 0 \quad ; \quad \langle 0|(\hat{\mathbf{E}}(\mathbf{r}, t))^2|0\rangle = \sum_{\ell} \mathcal{E}_{\ell}^2 \quad (111)$$

As expected, the mean value of the electric field is zero in vacuum, but its variance is non zero. We find here a new striking property of the quantum field, distinct from the existence of photons, which is the existence of *vacuum fluctuations*. They are nothing else than the zero point fluctuations of any quantum harmonic oscillator and give rise to a non-zero energy for the vacuum. Though these fluctuations exist in the ground state of the system, and therefore cannot transfer energy to any detector, they have many observable properties:

- As the Stark shift of atomic energy levels is proportional to the square of the applied electric field, vacuum fluctuations induce a non zero shift of the energy levels of any atom. In hydrogen, this shift turns out to be different in the $2S_{1/2}$ and $2P_{1/2}$ levels, which should be degenerate according to the Dirac theory of hydrogen. The small energy difference between these two levels, of pure quantum electrodynamical origin, has been observed by Lamb and Retherford, and is now called *the Lamb shift*. Recent experiments have given a relative agreement with the theoretical value of 10^{-6} , limited by the precision of the knowledge of the proton properties.
- The vacuum has also fluctuations of its magnetic field, which induce a random cyclotron motion of the position of an electron, and also fluctuations of its magnetic moment. These fluctuations affect the value of the *gyromagnetic factor* g of the electron, which should be equal to 2 according to the Dirac theory of hydrogen. The departure from the value 2, of the order of 10^{-3} , has been measured for the first time by Kusch. It has been more recently measured with a great accuracy, and is equal to the value calculated by quantum electrodynamics with a relative uncertainty of 10^{-8} .
- The vacuum has also fluctuations of its momentum, which may induce a random motion of objects put in vacuum and scattering the electromagnetic field. For symmetry reasons, the effect is zero on a plane mirror, for example, but it is non zero for two parallel mirrors, because

the structure of modes is different between the mirrors and outside. The net effect is an attractive force between the mirrors, which has also been measured: this is the *Casimir effect*.

If one calculates the value of vacuum fluctuations given in (111), one finds a divergence due to the high frequency part of the radiation field spectrum. This problem has been solved by using *renormalization procedures*, which have been introduced by Schwinger, Feynman and Tomonaga. We will not use it in this course, as we will deal mainly with the properties of a finite amount of modes which exclude this high frequency divergence.

The Wigner function of vacuum is

$$W(\bar{\alpha}) = \left(\frac{2}{\pi}\right)^N e^{-2|\bar{\alpha}|^2} \quad ; \quad W(\bar{R}) = \left(\frac{1}{2\pi}\right)^N e^{-|\bar{R}|^2/2} \quad (112)$$

where $|\bar{\alpha}|^2 = \sum_{\ell} |\alpha_{\ell}|^2$, $|\bar{R}|^2 = \sum_{\ell} (q_{\ell}^2 + p_{\ell}^2)$. It has a Gaussian shape of root mean square value $\Delta q_{\ell} = \Delta p_{\ell} = 1$. The vacuum is therefore a minimum state for the Heisenberg relation (97), with equal fluctuations on the two quadrature operators.

3.3 Gaussian states

3.3.1 Definition

They are states which are characterized by a Wigner function of gaussian shape, as for example the vacuum. They play an important role in quantum optics because they can be experimentally produced, and also because they have the following important property: *a pure quantum state the Wigner function of which is positive at all points is Gaussian* (Hudson-Piquet theorem). For such states therefore, the quasi-probability distribution becomes a well-behaved probability distribution, and they are the only quantum pure states to have this property.

They have the following general expression:

$$W(\bar{R}) = \left(\frac{1}{2\pi}\right)^N \frac{1}{\sqrt{Det\sigma}} \exp\left(-\left(\sum_{i,j=1}^{2N} (\bar{R} - \bar{R}_0)_i \sigma_{ij}^{-1} (\bar{R} - \bar{R}_0)_j\right)/2\right) \quad (113)$$

where \bar{R}_i is the i^{th} component of the point \bar{R}_i in the phase space $(q_1, p_1, \dots, q_N, p_N)$. A Gaussian state is therefore characterized by two quantities, the point in phase space \bar{R}_0 , and the $2N \times 2N$ matrix σ .

The Wigner function of the vacuum corresponds to the case where \bar{R} is zero and σ is the identity.

3.3.2 Properties

a-*Interpretation of \bar{R}_0 and σ*

>From the properties given in subsection(3.1.3), one easily shows that:

$$(\bar{R}_0)_i = \langle \hat{R}_i \rangle \quad (114)$$

$$\sigma_{ij} = \frac{1}{2} \langle \hat{R}_i \hat{R}_j + \hat{R}_j \hat{R}_i \rangle - \langle \hat{R}_i \rangle \langle \hat{R}_j \rangle \quad (115)$$

\bar{R}_0 is therefore the mean position in phase space, whereas σ is the *covariance matrix*, which contains all the variances and correlations between the quadrature components of the different modes. All these quantities can be experimentally measured using N homodyne detections.

b-*Generalization of the Heisenberg inequality*

In the single mode case, the variances are constrained by the Heisenberg inequality. This property extends to the N mode case: one can show that, in order to describe a physical quantum state, *the eigenvalues of the matrix $\sigma + i\Omega$ must be positive or zero*, Ω being the matrix of commutators $\Omega_{ij} = [\hat{R}_i, \hat{R}_j]/2i$.

c-*Purity*

The purity $P = \text{Tr}\rho^2$ of the quantum state has a simple expression, depending only on the covariance matrix σ :

$$P = \frac{1}{\sqrt{\text{Det}\sigma}} \quad (116)$$

For example in the single mode case, a pure Gaussian state has a covariance matrix of determinant 1: it is therefore a minimum state with respect to the Heisenberg inequality.

d-*Symplectic transforms*

Because of their linear character, they transform a quadratic form of the variables q_i, p_i into another quadratic form. Using property 6 of subsection

(3.1.3) , this implies that *the Gaussian character of a quantum state is preserved by symplectic transforms*. In particular a symplectic transformation transforms the vacuum state in a Gaussian state. Therefore any hamiltonian interaction which is bilinear in the different creation and annihilation operators produces Gaussian states from vacuum.

e-Williamson theorem

For any Gaussian state, there exists a symplectic transformation which transforms the Gaussian state into another one characterized by a *diagonal covariance matrix*, i.e. for which there is no correlation between the quadrature operators. In addition, the diagonal elements can be made be equal for the two quadratures of each given mode. The N variances obtained in such a way are called the "symplectic eigenvalues" of the Gaussian state, and characterize it in an intrinsic way. The generalized Heisenberg inequality (property a) requires that these N symplectic eigenvalues are not smaller than 1. The purity P is then nothing else than the inverse of the product of these N quantities. The purity is an invariant through symplectic transforms.

3.4 Classical and non-classical states of light

3.4.1 The semi-classical approach of quantum optics

Optical phenomena can be described by a theory which treats matter as a quantum object, having discrete energy levels, and light as a classical object, obeying the Maxwell equations, possibly affected by classical fluctuations when one wants to describe optical sources of finite temporal and spatial coherence. This approach is called the *semi-classical theory of light matter interaction*, and is remarkably successful: it is for example able to account for the photo-electric effect and the Compton effect, which are often considered as being a proof of the quantum character of light. Photons can be invoked to explain such phenomena, as Einstein and Compton did, but it was quickly realized by Dirac and Schrödinger that a theory involving classical fields and quantum atoms could explain them as well, just like the reflection of light on a mirror can be seen as particles bouncing on its surface, or waves obeying some boundary conditions. The semi-classical approach leads to the well-known Optical Bloch equations, which are able to explain a huge quantity of physical phenomena in quantum optics. They are able to account not only for the mean values of the measured quantities, but also for their fluctuations: in this

approach, when a weak classical single mode beam of light with a constant amplitude and phase interacts with a quantum photodetector, the sudden "clicks" recorded by the photodetector are not attributed to the arrival of a photon, but to the instantaneous character of the *quantum jump* undergone by an electron in the photodetector in presence of a constant interaction.

Some physicists have even questioned the practical utility of the quantum theory of light and claimed that all optical phenomena could be accounted for by a semi-classical theory. One now knows that it is not the case, and that many phenomena can only be explained by the full quantum theory of light and matter. We will call *non-classical states* the states of light which lead to predictions which cannot be made within the semi-classical theory, and classical states the others. Experimentally speaking, it is only in 1977, that physicists⁴ were able to produce a light state exhibiting "photon anti-bunching", which had no semi-classical explanation. Their experiment was the first to show in an unquestionable way that the quantum theory of light was actually indispensable⁵.

3.4.2 Simon criterion of non-classicality

Using the semi-classical theory, the photodetection process has the following property: if one shines a perfectly constant classical light beam on a photodetector, the resulting photocurrent has random fluctuations which are due to the statistical character of the quantum jumps. The corresponding photocurrent fluctuations, also called "shot noise", are given by:

$$\Delta i = \sqrt{2q \langle i \rangle \delta\nu} \quad (117)$$

where $\delta\nu$ is the detector bandwidth and $\langle i \rangle$ the mean photocurrent. The shot noise can also be simply written in photon units N (energy stored during the measurement time divided by $\hbar\omega$):

$$\Delta N = \sqrt{\langle N \rangle} \quad (118)$$

meaning that the occurrence of clicks in the photodetector is a Poisson process. If the beam is not perfectly coherent and has amplitude fluctuations, the photocurrent noise is of course larger than this value. The value (117)

⁴H.J. Kimble, M. Dagenais, L. Mandel, Phys. Rev. Letters **39** 691 (1977).

⁵A detailed account of bunching and anti-bunching effects can be found in A. Browaeys' lectures.

or (118) is usually called the *standard quantum limit* of photocurrent fluctuations⁶.

Using the full quantum theory of photodetection (section(2.3)), one can have quite different situations: for example, a number state, eigenstate of \hat{N} , incident on a photodetector, will yield a perfectly constant photocurrent, without any fluctuations. The limit between the semi-classical world and the pure quantum world is therefore set by the value of photocurrent fluctuations: *we will call "non-classical" a given state of light if there is a possibility of recording a photocurrent fluctuation on this state which is smaller than the standard quantum limit (117) or (118).*

Using this definition, it is easy to see that a classical multimode state (for which *all* the recorded photocurrent fluctuations are larger than the standard quantum limit) remains classical when it is submitted to any linear input-output transformation conserving the energy (the so-called "passive symplectic transformations"). Simon stated then the following more precise criterion of non-classicality:

A state of light is called non-classical when there exists a passive symplectic transformation which transforms it into a state in which the variance of a quadrature of at least one mode is below the standard quantum limit.

He showed that this criterion could be expressed in a simple mathematical way⁷:

A state is non-classical when the smallest of the covariance matrix eigenvalues is smaller than 1.

⁶It is indeed a quantum limit, but it is due to the quantum character of the detector, not to the quantum character of light.

⁷R. Simon, N. Mukunda, B. Dutta, Phys RevA **49** 1567 (1994).

4 SINGLE-MODE SYSTEMS

4.1 General remarks

4.1.1 Definition

It may happen in some situations that the state of the system expands only over the number states $|n_1\rangle \otimes \dots \otimes |n_\ell\rangle \dots$ with $n_1 = n_2 = \dots = n_{\ell-1} = n_{\ell+1} = \dots = 0$, n_ℓ being any integer. This occurs for example when one uses single mode lasers, which "fill" one mode with many photons, or when a resonant cavity selects a given mode. Though the state still belongs to the general Hilbert space defined on all the modes, and for example is characterized by non zero vacuum fluctuations in all the modes, we will call this kind of state a "single-mode state", and write it in the simplified way, omitting the zeros in all the other modes:

$$|\Psi\rangle = \sum_{n_\ell=0}^{\infty} c_{n_\ell} |n_\ell\rangle \quad (119)$$

We will here focus our attention on "single mode operators", i.e. to the restrictions of the observables to the single-mode Hilbert space, such as:

$$\begin{aligned} \hat{E}_\ell^{(+)}(\mathbf{r}, t) &= i\mathcal{E}_\ell \hat{a}_\ell e^{-i(\omega_\ell t - \mathbf{k}_\ell \cdot \mathbf{r})} \\ \hat{E}_{q\ell} &= i\mathcal{E}_\ell (\hat{a}_\ell - \hat{a}_\ell^\dagger) \quad \hat{E}_{p\ell} = \mathcal{E}_\ell (\hat{a}_\ell + \hat{a}_\ell^\dagger) \\ \hat{q}_\ell &= i(\hat{a}_\ell - \hat{a}_\ell^\dagger) \quad \hat{p}_\ell = \hat{a}_\ell + \hat{a}_\ell^\dagger \end{aligned} \quad (120)$$

4.1.2 Fresnel, or phasor, representation

The Wigner function associated with such systems is a real function of a single complex parameter α , or of two real parameters q, p . It can be represented as a 3D object in the space q, p, W . For example, figure (3) gives the Wigner function of a coherent state, which is a Gaussian state that we will study in subsection (4.2.2). Its maximum value gives \bar{R}_0 , i.e. the mean field, and the width of the q and p sections gives the r.m.s value of the corresponding quadrature operators. For any Gaussian state, the section of the Wigner function by a vertical plane making an angle ϕ with the q axis gives the probability distribution $P_\phi(x)$ of the operator $\hat{x}_\phi = \hat{a}^\dagger e^{i\phi} + \hat{a} e^{-i\phi}$ that can be measured by the homodyne technique described in subsection(2.5.3).

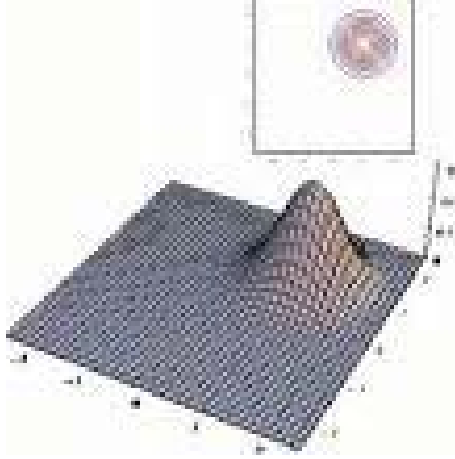


Figure 3: Wigner function of the single mode coherent state

In order to avoid manipulating 3D objects, one usually prefers to project the Wigner function on the q, p plane, and to only draw the point A of maximum W and the curve of constant quasi-probability $W(q, p) = e^{-1/2}$: the point A gives the mean complex field, within the propagation and scaling factor $\mathcal{E}_\ell e^{-i(\omega_\ell t - \mathbf{k}_\ell \cdot \mathbf{r})}$ and the size of the curve gives an idea of the field fluctuations: it is the *Fresnel, or phasor representation* of the single-mode state (see figure (4)). Whereas the cartesian coordinates in the q, p plane are related to the quadratures of the single mode field, the polar coordinates in this plane are related to its amplitude and to its phase.

4.2 A gallery of remarkable states

4.2.1 Number, or Fock, state

It is an eigenstate of the hamiltonian, and therefore gives stationary values to all observed quantities, including the electric field $\hat{E}_\ell(r, t)$. More precisely, one finds that in a number state:

$$\langle \hat{E}_\ell^{(+)}(\mathbf{r}, t) \rangle = 0 \quad \Delta \hat{E}_\ell^{(+)}(\mathbf{r}, t) = \Delta \hat{E}_{q\ell} = \Delta \hat{E}_{p\ell} = \sqrt{2n+1} \mathcal{E}_\ell \quad (121)$$

The mean field is zero, and the variances can be very far from their minimum value when n is large. Such a state is somehow similar to the state emitted by thermal sources (see subsection(4.2.3)) which has a random phase and

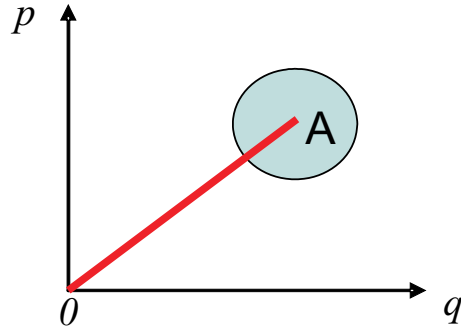


Figure 4: Simple phasor representation of a quantum state: the vector OA is related to the mean value of the complex field; the shaded area gives the region where the Wigner function takes significant values ($W(q, p) > e^{-1/2}$).

large quadrature fluctuations. But, in contrast to the thermal field, the fluctuations on the two quadratures are anti-correlated in such a way that such that there are no fluctuations on the field intensity $\hat{E}_{p\ell}^2 + \hat{E}_{q\ell}^2 = 2\mathcal{E}_\ell^2(2\hat{a}^\dagger\hat{a} + 1)$, which is proportional to the photon number operator.

Number states are simple to handle by theorists, but not by experimentalists, except for the vacuum state. The techniques of producing single photon states, described in A. Browaeys' lectures, are now well mastered. Some experiments have been able to produce number states with a few photons, mainly by conditional techniques. Nobody is so far able to produce a "macroscopic" number state with a large n value.

The Wigner function of the number state $|n\rangle$ is :

$$W_n(q, p) = \frac{1}{\pi} (-1)^n L_n(p^2 + q^2) e^{-(p^2 + q^2)/2} \quad (122)$$

$L_n(x)$ being the Laguerre polynomial of order n . For example, $L_0(x) = 1$, $L_1(x) = 1 - x$. Figure (5) shows the Wigner function of the single photon state, which has negative values for p and q values such that $p^2 + q^2 < 1$ (so that it can be orthogonal to the Gaussian state $|n = 0\rangle$ for example). The negative part of the Wigner function forbids us to interpret W as a well-behaved probability function. A number state is in this respect "more quantum" than a Gaussian state.

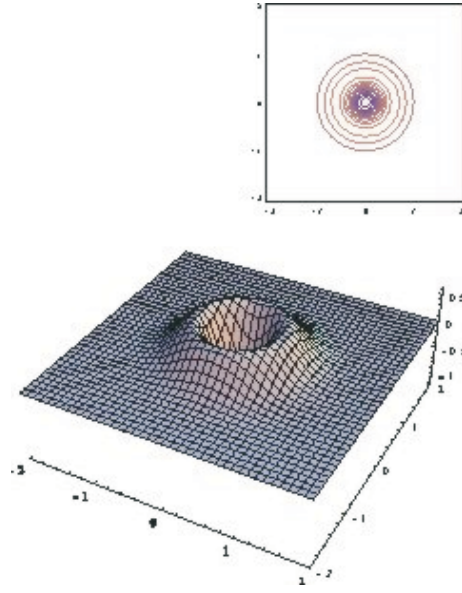


Figure 5: Wigner function of the single photon state

4.2.2 Coherent state

a-Definition

It can be easily shown that the spectrum of the annihilation operator \hat{a}_ℓ is the set of complex numbers, while there are no normalizable eigenstates for the creation operator \hat{a}_ℓ^\dagger . A coherent state (or Glauber state, or quasi-classical state) $|\alpha\rangle$ is defined as the eigenstate of \hat{a}_ℓ of eigenvalue α (α being any complex number), and therefore of the single mode complex electric field operator:

$$\hat{a}_\ell|\alpha\rangle = \alpha|\alpha\rangle \quad ; \quad \hat{E}_\ell^{(+)}(\mathbf{r}, t)|\alpha\rangle = \langle \hat{E}_\ell^{(+)}(\mathbf{r}, t) \rangle |\alpha\rangle \quad (123)$$

b-Main properties

Simple derivations show that in such a state:

$$\langle \hat{E}_\ell^{(+)}(\mathbf{r}, t) \rangle = i\mathcal{E}_\ell\alpha e^{-i(\omega_\ell t - \mathbf{k}_\ell \cdot \mathbf{r})} + \text{comp.conj} \quad \Delta \hat{E}_\ell^{(+)}(\mathbf{r}, t) = \Delta \hat{E}_{q\ell} = \Delta \hat{E}_{p\ell} = \mathcal{E}_\ell \quad (124)$$

The mean field has the expected classical spatio-temporal variation, but the field variances are independent of space-time position, independent of the eigenvalue α , and equal to the vacuum fluctuations⁸. It is, like the vacuum state, a minimum state with equal fluctuations on the two quadratures. Its expressions as a vector on the number state basis and its Wigner function are:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad ; \quad W_{\alpha}(q, p) = \frac{1}{2\pi} e^{-((p-p_0)^2 + (q-q_0)^2)/2} \quad (125)$$

with $q_0 + ip_0 = 2i\alpha$. The first expression shows us that the photon number probability in such a state is Poissonian, so that one can interpret this state as being composed of photons arriving at statistically independent times with a mean flow equal to $|\alpha|^2$. The second relation is similar to the Wigner function of vacuum (relation (112)), but displaced in the (q, p) plane around the point (q_0, p_0) (see figure (3)). Coherent states can be indeed obtained from the vacuum state by applying on it the displacement operator $\hat{D}(\lambda_{\ell} = \alpha)$ introduced in (94).

As coherent states are Gaussian states, with a positive Wigner function, they cannot be orthogonal. One shows more precisely that:

$$|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2} \quad (126)$$

Though they are never exactly orthogonal, they are almost orthogonal when the two eigenvalues differ by a number the modulus of which exceeds a few units.

In addition, one shows that when two coherent states $|\alpha_1^{in}\rangle$ and $|\alpha_2^{in}\rangle$ are incident on a beamsplitter, the output state is a tensor product of two other coherent states $|\alpha_1^{out}\rangle$ and $|\alpha_2^{out}\rangle$, *which are therefore not correlated*. The relations between the complex numbers α_i^{out} and the α_j^{in} are simply the relations relating the classical output complex amplitudes to the input ones on the beamsplitter.

Let us finally mention that one has the following decomposition of the identity $\hat{1}$ in terms of projectors over the coherent states:

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = \hat{1} \quad (127)$$

⁸The coherent state $|\alpha = 0\rangle$ is also the zero photon state or vacuum $|n = 0\rangle$. It is the only common eigenstate of the number operator and the annihilation operator

This decomposition does not mean that the coherent states form a basis of the single mode Hilbert space. The set of coherent states is "over-complete" and the decomposition is not unique. The identity operator can be also decomposed on a much smaller sub-set of $|\alpha\rangle\langle\alpha|$ projectors, for example the ones with a constant modulus.

c-The intense coherent state ($|\alpha|^2 \gg 1$)

One can easily show that:

$$\hat{a}_\ell^\dagger|\alpha\rangle = \alpha^*|\alpha\rangle + |\epsilon\rangle \quad ; \quad \langle\epsilon|\epsilon\rangle = 1 \quad (128)$$

and for the real electric field operator:

$$\hat{E}_\ell(\mathbf{r},t)|\alpha\rangle = \langle\hat{E}_\ell(\mathbf{r},t)\rangle|\alpha\rangle + |\epsilon'\rangle \quad ; \quad \langle\epsilon'|\epsilon'\rangle = \mathcal{E}_\ell^2 \quad (129)$$

For an intense coherent state the second terms in expressions (128) and (129) are negligible compared to the first ones. As a result, intense coherent states are "quasi-eigenstates" of the field operator. This property allows us to replace the field operator by its "quasi-eigenvalue" in many expressions, so that when the field is in such a state, the quantum theory of the system can be replaced by a *semi-classical theory* in which all the intense coherent fields are replaced by their classical time varying expressions. Another way to state the same property is to say that, in intense coherent states, the field fluctuations of the order of \mathcal{E}_ℓ , are negligible compared to the mean of the order of $|\alpha|\mathcal{E}_\ell$.

Intense coherent fields are thus "almost classical" states. This is not the case for low intensity coherent states ($|\alpha|^2 \approx 1$) for which quantum fluctuations play a very important part. For example, low intensity coherent states can be used for quantum cryptography.

d-Generation

These states are relatively easy to produce. Glauber has shown that classical oscillatory electrical currents, for example in HF antennas, radiate coherent states. A single mode laser operating well above its threshold produces an intense coherent state, at least for measurements which are made in a time much smaller than the Schawlow-Townes phase diffusion time. Any source with a non-zero mean field and non minimum quadrature noises evolves into a coherent state of small α when it is attenuated. When $|\alpha|^2 \ll 1$, it can be

written, up to second order in α :

$$|\alpha\rangle \approx (1 - \frac{\alpha^2}{2})|0\rangle + \alpha|1\rangle + \frac{\alpha^2}{2}|2\rangle + \dots, \quad (130)$$

This state is very different from the single photon state $|1\rangle$: it is mainly composed of vacuum, and also contains a non-negligible amount of photon pairs. Attenuated light has therefore not the same properties as single photons.

4.2.3 Statistical superpositions of coherent states

They are described by density matrices of the form:

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \quad (131)$$

$P(\alpha)$ being the classical, positive, probability distribution of having a coherent state with the value α .

When $|\alpha|$ is large, ρ describes a genuine classical state with a finite classical temporal coherence (in the classical optics meaning), where the phase and amplitude fluctuations are of classical origin.

The most important example of such a state is the *thermal state* ρ_T , that one obtains when the radiation field is at equilibrium with a reservoir at temperature T . Its general expression is $\rho_T = \exp(-\hat{H}_R/k_B T)/Z$, where the partition function Z is a normalizing factor. One can show that this implies ρ_T is of the form (131) with:

$$P(\alpha) = \frac{1}{\pi\bar{n}} e^{-|\alpha|^2/\bar{n}} \quad (132)$$

\bar{n} being the mean number of thermal photons ($\bar{n}^{-1} = e^{\hbar\omega_\ell/k_B T} - 1$). A thermal state can thus be seen as a statistical superposition of coherent states with a Gaussian statistics.

The thermal state can be described in the number state basis and as a Wigner function by the following expressions:

$$\rho_T = \bar{n}^{-1} \sum_n e^{-(n+1)\frac{\hbar\omega_\ell}{k_B T}} |n\rangle\langle n| \quad W(q, p) = \frac{1}{2\pi(2\bar{n} + 1)} \exp\left(-\frac{p^2 + q^2}{2(2\bar{n} + 1)}\right) \quad (133)$$

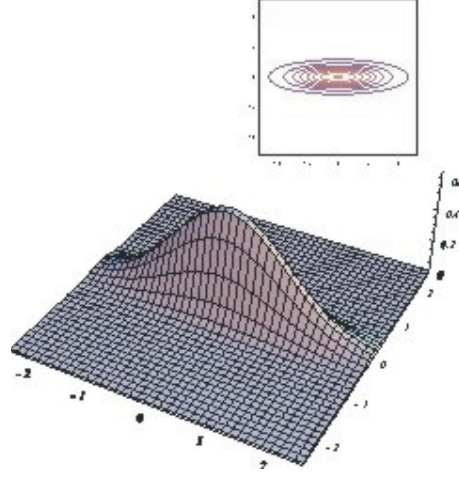


Figure 6: Wigner function of a squeezed vacuum state

The density matrix is diagonal in the number state basis, and the coefficients form a geometrical series. The Wigner function is Gaussian and symmetric. In such a state, the mean field is zero, and the variance of any quadrature is $(2\bar{n} + 1)\mathcal{E}_\ell^2$, just like in a number state. But the Wigner function is always positive, which is not the case for a number state. From (116) one finds that the purity P of a thermal state is $P = (2\bar{n} + 1)^{-1}$. Therefore, if $\bar{n} \ll 1$, i.e. if $\hbar\omega_\ell \gg k_B T$, which is the case in the optical range at room temperature, the thermal state is very close to a pure state, namely the vacuum state.

4.2.4 Squeezed states

a-*Definition and main features*

Let us introduce the operator

$$\hat{A}_S = \hat{a}_\ell \cosh S + \hat{a}_\ell^\dagger \sinh S \quad (134)$$

It obeys the commutation relation of an harmonic oscillator annihilation operator $[\hat{A}_S, \hat{A}_S^\dagger] = 1$. The squeezed states $|\alpha, S\rangle$ are the coherent states of this annihilation operator:

$$\hat{A}_S |\alpha, S\rangle = \alpha |\alpha, S\rangle \quad (135)$$

We will call "squeezed vacuum" the state $|\alpha = 0, S\rangle$, which has the following expression in the number state basis:

$$|\alpha = 0, S\rangle = \frac{1}{\sqrt{\cosh S}} \sum_n (\tanh S)^n \frac{\sqrt{2n!}}{n!} |2n\rangle \quad (136)$$

It only expands on the even number states, and is sometimes called a "two-photon coherent state". Its Wigner function has the following expression:

$$\begin{aligned} W(\alpha) &= \frac{2}{\pi} \exp(-2|(\alpha - \alpha_0) \cosh S + (\alpha^* - \alpha_0^*) \sinh S|^2) \quad (137) \\ W(q, p) &= \frac{1}{2\pi} \exp(-(q - q_0)^2 e^{-2S}/2 - (p - p_0)^2 e^{2S}/2) \end{aligned}$$

It is therefore a Gaussian state, the variances of which are different on the two quadrature. It is depicted in figure (6). More precisely, it is easy to see that:

$$\begin{aligned} \Delta E_p &= \mathcal{E}_\ell e^{-S} & \Delta E_q &= \mathcal{E}_\ell e^S & \Delta E_p \Delta E_q &= \mathcal{E}_\ell^2 \\ \Delta^2 \hat{E}_\ell(r) &= \mathcal{E}_\ell^2 (e^{2S} \cos^2 \mathbf{k}_\ell \cdot \mathbf{r} + e^{-2S} \sin^2 \mathbf{k}_\ell \cdot \mathbf{r}) \end{aligned} \quad (138)$$

Squeezed states are minimum states, in which the electric field fluctuations are alternatively greater and smaller than the vacuum fluctuations, on the distance of half the wavelength. When $S \rightarrow \pm\infty$, the squeezed state goes to an eigenstate of $\hat{E}_{p\ell}$ or $\hat{E}_{q\ell}$. These states are very useful to improve measurements made on a single quadrature, for example in interferometry (see A. Heidmann's lectures).

They can be generalized if one consider now the operator $\hat{A}_{S,\phi} = \hat{a}_\ell e^{i\phi} \cosh S + \hat{a}_\ell^\dagger e^{-i\phi} \sinh S$, which still obeys the commutation relation of an annihilation operator. One introduces a generalized squeezed state $|\alpha, S, \phi\rangle$ as an eigenstate of $\hat{A}_{S,\phi}$ with the eigenvalue α . These states are also Gaussian states for which:

$$\begin{aligned} \Delta \hat{x}_\phi &= \mathcal{E}_\ell e^{-S} & \Delta \hat{x}_{\phi+\frac{\pi}{2}} &= \mathcal{E}_\ell e^S & \Delta \hat{x}_\phi \Delta \hat{x}_{\phi+\frac{\pi}{2}} &= \mathcal{E}_\ell^2 \\ \Delta^2 \hat{E}_\ell(r) &= \mathcal{E}_\ell^2 (e^{2S} \cos^2(\mathbf{k}_\ell \cdot \mathbf{r} + \phi) + e^{-2S} \sin^2(\mathbf{k}_\ell \cdot \mathbf{r} + \phi)) \end{aligned} \quad (139)$$

where $\Delta \hat{X}_\phi$ is the rotated quadrature operator introduced with the homodyne detection (subsection (2.5.2)). The states $|\alpha, S, \phi\rangle$ are therefore the usual squeezed states rotated by an angle ϕ in the Fresnel plane.

b-*Generation*

The input-output relations have the following important property, that can be easily derived: if the input state is an eigenstate of the input operators, then the output state is an eigenstate of the output operator. So starting from the vacuum state, or from any coherent state, an input-output relation of the form (134) will produce a squeezed state at its output. This is the case for the *degenerate parametric interaction*, obeying the input-output relation (84) with $\hat{E}_{1,in}^{(+)} \equiv \hat{E}_{2,in}^{(+)}$, i.e. when the signal and idler modes are identical. Experimentally speaking, one first needs to use a parametric crystal in the right phase matching conditions to achieve degenerate parametric interaction. Then a first possibility is to pump it with an intense pulsed laser: one gets in this way squeezed pulses of light. The second solution is to use a c.w. less intense pump laser and a cavity which is resonant on the common signal-idler mode and enhances the non linear interaction. One builds in this way a (degenerate) Optical Parametric Oscillator (OPO), likely to produce c.w. intense beams, like a laser, above some pump power threshold. Just below the oscillation threshold, the system produces highly squeezed states. In both configurations, the exact degeneracy conditions are more easily obtained when one injects at the input of the crystal a small coherent state at half the pump frequency, which is parametrically amplified (in a phase-sensitive way) by the non-linear crystal.

c-*Small fluctuation approximation*

In many experimental situations, the input fields are "intense" enough so that their mean value is much larger than \mathcal{E}_ℓ . In this case, the field fluctuations are negligible compared to the mean fields. Let us introduce a *fluctuation operator*:

$$\delta\hat{E} = \hat{E} - \langle \hat{E} \rangle \quad (140)$$

where E stands for any field, annihilation or creation operator. It is in this case a good approximation to *linearize the input-output relations* of any non-linear interaction around the mean values of the field operators. One ends up with input-output relations of the form:

$$\delta\hat{E}_\ell^{(+out)} = \sum_{\ell'} \left(A_{\ell'}^{\ell'} \delta\hat{E}_{\ell'}^{(+in)} + B_{\ell'}^{\ell'} \delta\hat{E}_{\ell'}^{(-in)} \right) \quad (141)$$

which, by construction, obey the canonical commutation relations. These are therefore symplectic transformations, which contain the squeezing transformations. In particular, third order non-linear effects like the optical Kerr

effect, which leads to an intensity-dependent index of refraction in media such as optical fibers, yield to an input-output relation that is quadratic in the annihilation and creation operators. When this relation is linearized around the mean value, i.e. when the Kerr medium is submitted to intense fields, one gets a squeezing transformation. For example Kerr media such as glass are often used experimentally to generate squeezed states by sending a coherent state in a long optical fiber. But when the Kerr media are pumped by a low intensity coherent state, or when the Kerr effect is huge, the linearization approximation drops, and the Wigner function of the output state is no longer Gaussian.

Let us note that at this level of approximation, intensity fluctuations and quadrature operator fluctuations are proportional, because:

$$\delta(\hat{a}^\dagger\hat{a}) \simeq A(\delta\hat{a}^\dagger e^{i\phi} + \delta\hat{a}e^{-i\phi}) = A\delta\hat{x}_\phi \quad (142)$$

where $\langle \hat{a} \rangle = Ae^{i\phi}$. Intensity fluctuations are in particular proportional to the fluctuations $\delta\hat{q}$ and $\delta\hat{p}$ when ϕ is zero or $\pi/2$. More generally, expression (142) is very similar to expression (87) of the signal in homodyne detection. A direct intensity measurement, at the small fluctuation approximation, is thus a kind of "self-homodyne technique", the local oscillator being the mean field itself.

4.2.5 Schrödinger cats

Generalizing the well known example given by Schrödinger to illustrate the paradoxical character of quantum superpositions, quantum opticians call *Schrödinger cat states* quantum states of light which are linear superposition of macroscopically distinguishable light states. As we have seen in section (4.2.2), intense coherent states are indeed the quantum states which are able to describe classical fields. Among others, Schrödinger cat states can be of the form:

$$|\Psi_{cat}^1\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle \pm |0\rangle) \quad ; \quad |\Psi_{cat}^2\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle \pm |-\alpha\rangle) \quad (143)$$

with $|\alpha| \gg 1$. The first one is the "superposition of light and obscurity", close to the superposition of a dead and alive cat. The second one, of more wavelike character, is the strange superposition of two classical waves of opposite amplitudes. These states are highly quantum and have Wigner

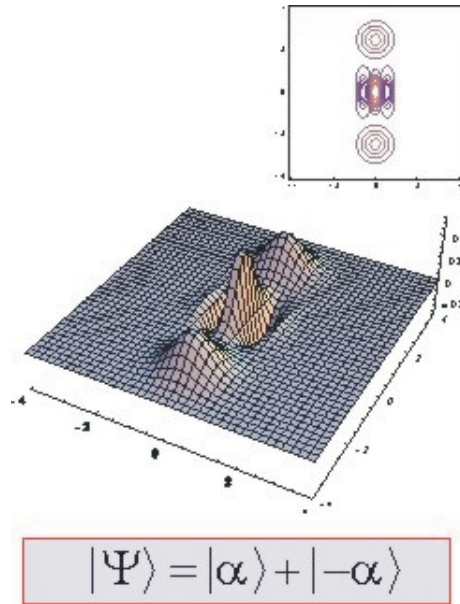


Figure 7: Wigner function of a Schrödinger kitten state

functions consisting of the two Gaussian peaks of the coherent states, plus large interferences fringes in between which take large negative values (see figure (7)).

Actually such states are very fragile when they are submitted to losses, which induce strong decoherence effects: the loss of the energy of a few photons is able to transform them into mixed states such as:

$$\rho_{cat}^1 = \frac{1}{2}(|\alpha\rangle\langle\alpha| + |0\rangle\langle 0|) \quad ; \quad \rho_{cat}^2 = \frac{1}{2}(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|) \quad (144)$$

Their Wigner functions are now positive and contain two Gaussian peaks but no fringes. "Schrödinger kitten states"⁹ of the same form (143) but with $|\alpha| \simeq 1$ are somewhat more easily to handle, and have actually been generated in a conditional way by different groups in the world.

4.2.6 Single mode non-classical states of light

We have seen in this section that, as far as photocurrent fluctuations are concerned, number states and squeezed states lead to fluctuations smaller

⁹See J. Wenger's lecture

than the standard quantum limit, whereas coherent states are at the standard limit, and statistical superpositions of coherent states lead to fluctuations larger than this limit. Number states and squeezed states are therefore in the category of non-classical states. One can show that it is also the case for Schrödinger cat states.

Coherent states are just at the boundary between classical and non-classical states: this is why they are called "quasi-classical" states. They are often used in experiments to set the experimental value of the standard quantum limit.

5 CORRELATIONS IN BIPARTITE SYSTEMS

We now consider a *bipartite system*, which is formed of two separated parts, called 1 and 2, that have interacted in the past and are spatially well separated so that one can make measurements on the field in part 1 without physically perturbing the part 2. In the majority of cases studied below these two parts will be two well-separated modes of the radiation field. Many conclusions are also valid in the case where the bi-partite system is formed of two separated sets of modes.

5.1 Characterization of correlations

Let us first recall some general properties of correlations between the measurements of two physical quantities A and B , which are performed on a system which is not necessarily bipartite.

5.1.1 Correlation function

If the measurement of the quantities A and B on the system has yielded values a and b , one obtains a point of coordinates a, b in a two-dimensional space. If one reproduces many times the same system and measure again A and B on each one, one obtains many points which form a "cloud". When these points are well-aligned, the measurement of A for example gives a lot of information about the value of B : there is a strong correlation between the two quantities. The amount of correlation is characterized by the *correlation function* C_{AB} :

$$C_{AB} = \frac{1}{2} \langle AB + BA \rangle - \langle A \rangle \langle B \rangle = \frac{1}{2} \langle \delta A \delta B + \delta B \delta A \rangle \quad (145)$$

where $\delta A = A - \langle A \rangle$. This definition applies to classical fluctuating quantities (in this case $AB = BA$, and the mean $\langle \rangle$ is taken on the classical statistical ensemble) and also to quantum observables \hat{A} and \hat{B} . In both cases, the Cauchy-Schwartz inequality tells us that:

$$C_{AB}^2 \leq \Delta^2 A \Delta^2 B \quad (146)$$

one can then define a *normalized correlation coefficient* c_{AB} equal to:

$$c_{AB} = \frac{C_{AB}}{\Delta A \Delta B} \quad (147)$$

which varies between -1 (perfect anti-correlation) to $+1$ (perfect correlation through 0 (absence of correlation)).

Perfect correlations (or anti-correlations) between distant objects exist in the classical world as well as in the quantum world. For example the values of the intensities of two electrical currents obtained by dividing a given current in two parallel derivations are proportional and therefore are perfectly correlated classical quantities: measuring the intensity of the current in part 1 allows us to immediately predict the intensity in the other wire without interacting with it: taking advantage of perfect correlations is the most easy way to perform a *Non Demolition measurement* on a system.

5.1.2 Conditional variance

When the correlation between A and B is not perfect, measuring A does not allow us to perfectly know B , but it reduces our uncertainty on the possibly measured values of B . The reduced uncertainty on B knowing A is characterized by the *conditional variance* $\Delta^2(B|A)$ given by:

$$\Delta^2(B|A) = \Delta^2 B (1 - c_{AB}^2) = \Delta^2 B - \frac{C_{AB}^2}{\Delta^2 A} \quad (148)$$

$\Delta^2(B|A)$ is always smaller than $\Delta^2 B$ and is zero when the (anti-)correlation is perfect. Let us introduce the "corrected" B value, equal to $\delta B_\lambda = \delta B - \lambda \delta A$, obtained by correcting the fluctuations of δB by some amount of the fluctuations of the correlated quantity A . It is easy to show that the conditional variance $\Delta^2(B|A)$ is the minimum value of $\Delta^2 B_\lambda = \langle \delta^2 B_\lambda \rangle$ for all the possible choices of the contamination factor λ .

5.1.3 Correlation between the two beams generated by a beam-splitter

Let us consider the simple example of a 50% beamsplitter: in classical optics the two output beams obtained by splitting a given input beam have equal amplitudes, and are therefore perfectly correlated. This no longer the case at the quantum level, when one takes into account the vacuum fluctuations entering the unused input port of the beamsplitter. Using the beamsplitter quantum input-output relations (82) with $r = t = 1/\sqrt{2}$, one easily finds the following expression for the normalized correlation coefficient between

quadrature measurements performed on the output beams 1 and 2:

$$c_{12} = \frac{F^{in} - 1}{F^{in} + 1} \quad (149)$$

where the "input Fano factor" F^{in} is the ratio between the variance of the measured input beam quadrature and the vacuum fluctuation variance \mathcal{E}_ℓ^2 ¹⁰. One finds an almost perfect correlation if the input field is very noisy ($F^{in} \gg 1$: vacuum fluctuations are then negligible), and no correlations when the input beam is in a coherent state ($F^{in} = 1$: as already seen in section(4.2.2) a beamsplitter produces from two input coherent states a tensor product of two output coherent states). One finds perfect anticorrelations for an input number state or perfectly squeezed state ($F^{in} = 0$).

5.2 Classical and quantum correlations

The existence of strong, or even perfect, correlations or anti-correlations does not ensure the "quantum character" of their origin. We need therefore to have criteria allowing us to decide the amount of such quantum character revealed by a given correlation.

5.2.1 Heisenberg inequality for a correlation

When \hat{A} and \hat{B} are quantum observables, one has the following generalized Heisenberg inequality¹¹, valid for any quantum state of the system:

$$\Delta^2 A \Delta^2 B \geq \frac{1}{4} | \langle [\hat{A}, \hat{B}] \rangle |^2 + C_{AB}^2 \quad (150)$$

This inequality implies the usual Heisenberg inequality when one forgets the C_{AB}^2 term, and the Cauchy-Schwartz inequality when one forgets the commutator term. It yields a new Heisenberg inequality, valid for correlated measurements, which deals with the variance of A and the conditional variance of B knowing A :

$$\Delta^2 A \Delta^2(B|A) \geq \frac{1}{4} | \langle [\hat{A}, \hat{B}] \rangle |^2 \quad (151)$$

¹⁰This quadrature can be the amplitude quadrature, and therefore one can extend the present discussion to intensity measurements provided that the small fluctuation approximation is valid.

¹¹See for example J.J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley

In addition, it gives an upper limit to the normalized correlation coefficient:

$$c_{AB}^2 \leq 1 - \frac{|\langle [\hat{A}, \hat{B}] \rangle|^2}{4\Delta^2 A \Delta^2 B} \quad (152)$$

As a result, two non-commuting observables cannot be perfectly (anti-)correlated. In contrast, measurements performed on the two separated parts of a bipartite system are always associated with commuting observables \hat{A}_1 and \hat{B}_2 . In this case, perfect quantum (anti-)correlations are possible.

5.2.2 Non-classical correlation: "gemellity"

Let us consider an optical system producing two output beams. We want to know whether the correlation that we observe between measurements performed on these two beams can be described or not within the semi-classical approximation (classical fluctuating fields and quantum detectors) mentioned in section (3.4.2). It is easy to see that the mixing of the two beams on a beamsplitter does not change the classical or non-classical character of a beam. We will therefore say that two beams are correlated in a non-classical way if, by appropriately mixing them, one obtains a single mode non-classical beam, i.e. a sub-Poissonian beam. This statement is nothing else than the general Simon definition of a non-classical state of light given in section (3.4.2) applied to the two-mode case. One can then use the Simon criterion to characterize it: the smallest of the covariance matrix eigenvalues must be smaller than 1. When one calculates the minimum eigenvalue of the 4×4 covariance matrix and impose that it must be smaller than 1, one obtains a general criterion for a non-classical correlation that has a complicated expression, difficult to interpret¹². It is more interesting to write the criterion in the simple "balanced" form of the covariance matrix. In this particular case, the variances are equal in the modes 1 and 2 for each quadrature ($\Delta E_{q1} = \Delta E_{q2}$; $\Delta E_{p1} = \Delta E_{p2}$), and there are no correlations between the q and p quadratures, inside the two modes and between the two modes. If for example the smallest eigenvalue corresponds to the q quadrature, the Simon criterion of non-classicality of section(3.4.2) becomes:

$$G = F_q(1 - |c_{E_{q1}E_{q2}}|) < 1 \quad (153)$$

¹²N. Treps, C. Fabre, *Criteria of quantum correlation in the measurement of continuous variables in optics* Laser Physics **15** 187 (2005)

where G is the so-called "*gemellity*" of the two beams 1 and 2, and $F_q = \Delta^2 E_{q1} / \mathcal{E}_\ell^2$ is the common Fano factor of the two beams on the q quadrature. A gemellity smaller than 1 implies that the observed correlations between the two beams have no semi-classical description. This criterion implies a lower limit for the normalized correlation coefficient between the q -quadratures of the two beams in a non-classical system:

$$|c_{E_{q1}E_{q2}}| > 1 - \frac{1}{F_q} \quad (154)$$

In order to be non-classical, the correlation must be close enough to 1, especially when the two beams are noisy. Inversely, any correlation between two beams at the standard quantum level ($F_q = 1$) is of quantum origin.

There is another expression of the gemellity, always in the balanced case of equal variances for the two correlated beams:

$$G = \frac{\Delta^2(E_{q1} - E_{q2})}{2\mathcal{E}_\ell^2} \quad (155)$$

meaning that the fluctuations of the quadrature difference observable $(\hat{E}_{q1} - \hat{E}_{q2})/\sqrt{2}$ are below the standard quantum limit. This is the reason why non-classical correlations of this kind are sometimes called *difference squeezing*.

In the case when one is interested in the amplitude quadratures of the two modes, such non-classical beams are called *twin beams*, as they have almost identical intensity fluctuations. The gemellity G can be easily measured in the case of twin beams: one just has to subtract the photocurrents coming from the two photodiodes directly measuring the intensity fluctuations of the two beams.

Let us go back to the case of two beams produced by splitting an incident beam of Fano factor F_q^{in} on a 50% beamsplitter. We are in the balanced case, because the two output beams have equal mean intensities and fluctuations. It is easy to show that the gemellity has the following expression in this case:

$$F_q^{in} > 1 : G = 1 \quad ; \quad F_q^{in} \leq 1 : G = F \quad (156)$$

We have here an example of what we have already noticed: a classical beam ($F_q^{in} > 1$) gives on a beamsplitter two other classical beams, with classical correlations ($G = 1$). The important feature is that $G = 1$ whatever the noise on the classical input beam. In order to measure the standard quantum level for the gemellity, a widely used technique is just to measure the gemellity between such two beams.

5.2.3 Criterion of QND correlation

As we have already stated, correlations allow us to perform Non Demolition measurements. But in which respect is this ND measurement a "QND" measurement (Quantum Non Demolition measurement) ? It is when a measurement on beam 2 allows us to obtain some information about the *quantum fluctuations* of beam 1. This information can then be used in an active feed-forward correction scheme of the fluctuations of beam 1, with the help for example of an electro-optic modulator, and if there are QND correlations, the resulting beam should be below the standard quantum limit. As we have seen in section (5.1.2), the optimum correction scheme allows us to reach the conditional variance of beam 1 fluctuations, knowing beam 2 fluctuations.

We will therefore say that a given correlation between the quadratures E_{q1} and E_{q2} is a QND correlation if one has:

$$\Delta^2(E_{q1}|E_{q2}) < \mathcal{E}_\ell^2 \quad (157)$$

Using expression (148) of the conditional variance, this implies the following inequality for the Fano factor and the normalized correlation:

$$V_q = F_q(1 - c_{E_{q1}E_{q2}}^2) < 1 \quad (158)$$

An expression close to the non-classicality criterion (153), but different. The QND criterion implies a lower limit for the normalized correlation coefficient :

$$|c_{E_{q1}E_{q2}}| > \sqrt{1 - \frac{1}{F_q}} \quad (159)$$

In order to be QND, the correlation must be even closer to 1 than the value needed to be non-classical. More precisely, one shows that there is the following order between the gemellity G and the normalized conditional variance V_q :

$$G \leq V_q \leq 2G \quad (160)$$

The presence of a QND correlation is a stronger quantum effect than the non-classical character of the beams. In addition, a gemellity smaller than 0.5 is enough to ensure that the two beams are QND correlated.

Finally, one easily shows that the normalized conditional variance between two beams produced by a 50% beamsplitter is equal to $2F_q^{in}/(F_q^{in} + 1)$. It is smaller than 1 as soon as F_q^{in} is smaller than 1: splitting a squeezed beam on a 50% beamsplitter produces QND correlations.

5.3 Entangled states

5.3.1 Definition and main properties

a-*Pure state*

Let us consider the most general bi-modal pure state, described by the state vector:

$$|\Psi\rangle = \sum_{n_1, n_2} c_{n_1, n_2} |n_1, n_2\rangle \quad (161)$$

There is a very useful theorem due to Schmidt, which establishes that such a state can always be written in the following way:

$$|\Psi\rangle = \sum_{i=1}^S \alpha_i |u_i\rangle \otimes |v_i\rangle \quad (162)$$

where $|u_i\rangle$ and $|v_i\rangle$ are vectors belonging respectively to the Hilbert space of mode 1 and mode 2 and forming two orthonormal sets, and the α_i are S nonzero complex numbers such that $\sum_i |\alpha_i|^2 = 1$. Expression (162) is called the Schmidt decomposition of the bimodal vector $|\Psi\rangle$. Though this decomposition is not unique, the number S , called the Schmidt number, is unique, and is linked to the "complexity" of the considered quantum state.

- If $S = 1$, then $|\Psi\rangle = |u_1\rangle \otimes |v_1\rangle$: the state is factorized. If \hat{A}_1 and \hat{B}_2 are any "local" observables acting respectively only on mode 1 and 2, then $\langle \hat{A}_1 \hat{B}_2 \rangle = \langle \hat{A}_1 \rangle \langle \hat{B}_2 \rangle$, and $C_{A_1, B_2} = 0$: there are no correlations between any local observables in such a state.
- If $S > 1$, then the state cannot be factorized and is called an "entangled state". Correlations may now occur between measurements performed on parts 1 and 2. So, for a pure state, there is a very simple entanglement criterion: a pure state is entangled if and only if there exists local measurements on parts 1 and 2 which exhibit some degree of correlation. Entanglement and correlation are therefore synonymous, whatever the kind of correlation, classical, non-classical or QND which is observed in the system. Furthermore, observables which are diagonal in the basis $|u_i\rangle$ or $|v_i\rangle$, called "Schmidt observables", of the form:

$$\hat{A}_1 = \sum_{i=1}^S a_i |u_i\rangle \langle u_i| \quad ; \quad \hat{B}_2 = \sum_{i=1}^S b_i |v_i\rangle \langle v_i| \quad (163)$$

where a_i and b_i are any real numbers which are all different, give a correlation coefficient C_{A_1, B_2} of 1 in state $|\Psi\rangle$, because the measurement of the value b_k of B_2 will project the initial state on $|u_k\rangle \otimes |v_k\rangle$, giving without any uncertainty the value a_k to the measurement of A_1 .

-b *Mixed state*

A density matrix ρ is called separable if one can write it in the following form:

$$\rho = \sum_{i=1}^k p_i |u_i\rangle \otimes |v_i\rangle \langle u_i| \otimes \langle v_i| \quad (164)$$

p_i being real positive numbers with $\sum_i p_i = 1$. Such a state is therefore a statistical mixture of factorized pure states.

If one cannot write ρ as (164), then the mixed state is named "entangled" or "non-separable".

It is easy to see that one has perfect correlations between Schmidt observables \hat{A}_1 and \hat{B}_2 of the form (163) in a separable state: correlations and entanglement are no longer synonymous for a mixed state.

5.3.2 Examples

a- *Pure twin beams*

They have the following Schmidt decomposition:

$$|\Psi_{twin}\rangle = \sum_{i=1}^{\infty} c_n |n\rangle \otimes |n\rangle = \sum_{i=1}^{\infty} c_n |n, n\rangle \quad (165)$$

As the photon number operators $\hat{a}_1^\dagger \hat{a}_1$ and $\hat{a}_2^\dagger \hat{a}_2$ are Schmidt observables of the form (163), they are perfectly correlated in such a state whatever the value of c_n . The state is formed of "twin photons" which were born at the same time in the source and have been separated at the output. This can be seen also on the property that $|\Psi_{twin}\rangle$ is an eigenstate of the intensity difference operator $\hat{N}_1 - \hat{N}_2 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2$ with the eigenvalue 0, so that the variance $\Delta^2(N_1 - N_2)$ vanishes in such a state: the gemellity G is also zero in this state.

b- *Mixed twin beams*

They are described by the following separable density matrix:

$$\rho_{twin} = \sum_{i=1}^{\infty} |c_n|^2 |n, n\rangle \langle n, n| \quad (166)$$

As stated in the previous section, there is also a perfect correlation between the photon numbers \hat{N}_1 and \hat{N}_2 in the two modes, and no fluctuations on the intensity difference $\hat{N}_1 - \hat{N}_2$.

As the normalized correlation $c_{\hat{N}_1, \hat{N}_2}$ is equal to 1, the criteria (153) and (158) tell us that this separable state is non-classical (being a statistical superposition of non-classical number states), and that it allows us to make perfect QND measurements of the beam intensity.

This shows that in addition to entangled states, one can find separable states which are non-classical states of considerable physical interest.

c- Two-mode squeezed states

Let us go back to the situation of parametric mixing that we have introduced in section (2.4.4), described in the operatorial point of view by the input-output relation (85). The corresponding transformation for the states is:

$$|\psi^{out}\rangle = e^{S(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)} |\psi^{in}\rangle \quad (167)$$

corresponding to an hamiltonian evolution in which photons in the signal and idler modes are created or destroyed simultaneously. If the input state is the vacuum, then one can show that:

$$|\psi^{out}\rangle = \frac{1}{\cosh S} \sum_n (\tanh S)^n |n, n\rangle \quad (168)$$

which is indeed a pure twin beam.

When the parametric crystal is pumped by a cw laser of a few Watts, the value of S is small so that $|\psi^{out}\rangle \simeq |0, 0\rangle + S|1, 1\rangle$: $|\psi^{out}\rangle$ is a "twin-photon state", a very interesting quantum state which has been the subject of many investigations (see A. Browaeys lectures). When the crystal is inserted in a resonant cavity, one has an OPO, similar to the one described in section (4.2.3.b) (but simpler, as we no longer require the signal and idler mode frequencies to be identical). Below the oscillation threshold of the device, the OPO produces a two-photon squeezed state like (168) with $S \rightarrow \infty$ when the pump intensity approaches the threshold value. Above the oscillation threshold one gets intense pure twin beams of the form (165) with $|c_n|^2$ peaked for very high n values. Experimentally speaking the best measured value of the gemellity is $G \simeq 0.1$.

5.3.3 Entanglement criteria

We will just consider here the case where each part consists of a single mode field and give only a few examples of such criteria, as new ones are published almost monthly. Entanglement criteria are even more difficult to find in the general bipartite case.

a- *Positive partial transpose*

This general criterion, introduced by Peres and Horodecki, concerns the partial transpose ρ^{T1} of the bipartite density matrix ρ defined by:

$$\langle n_1, n_2 | \rho^{T1} | n'_1, n'_2 \rangle = \langle n'_1, n_2 | \rho | n_1, n'_2 \rangle \quad (169)$$

It states that the density matrix ρ is separable if and only if all the eigenvalues of the partial transpose ρ^{T1} are positive or zero. This criterion is useful from a mathematical point of view, but it is not operational as one has no experimental access to the partial transpose of the state under study.

b- *Criterion for Gaussian states*

The problem of finding a criterion is simplified if one restricts oneself to a sub-space of all possible quantum states, for example the subset of gaussian states described in section (3.3). These states are characterized by a 4×4 covariance matrix σ that one can write as made of four 2×2 sub-matrices of correlation and variances:

$$\sigma = \begin{bmatrix} V_1 & C_{12} \\ C_{12}^T & V_2 \end{bmatrix} \quad (170)$$

Adesso and Illuminati have given an entanglement criterion in terms of the global purity $P = Tr\sigma^2 = 1/\sqrt{Det\sigma}$ and the marginal purities $P_i = 1/\sqrt{DetV_i}$. They have shown that the state is entangled if the following inequality holds:

$$P > \frac{P_1 P_2}{\sqrt{P_1^2 + P_2^2 - P_1^2 P_2^2}} \quad (171)$$

and that the state is separable if the following inequality holds:

$$P < \frac{P_1 P_2}{P_1 + P_2 - P_1 P_2} \quad (172)$$

an narrow intermediate region existing for P in which the entanglement cannot be assessed by a criterion based only on the purities.

Duan et al.¹³ have given a necessary and sufficient criterion based on the covariance matrix when it is written in the so-called "normal form", i.e. after the system has undergone a specific symplectic transformations acting separately on the two parts, which preserve the entanglement property.

All these criteria can be used to assess the entanglement of a state of light produced in an experiment, but only when one is able to record by homodyne techniques the fluctuations of the two quadratures of the two modes and derive from them the values of all the elements of the covariance matrix σ . But it is not very practical as it requires many measurements on the system.

c- Operational criteria for Gaussian states

They allow us to ascertain by only two measurements that a given Gaussian state is entangled. They are only necessary conditions, so that some entangled states are beyond reach of these criteria. They are based on the measurements of the two gemellities:

$$G_-^p = \frac{\Delta^2(E_{p1} - E_{p2})}{2\mathcal{E}_\ell^2} \quad ; \quad G_+^q = \frac{\Delta^2(E_{q1} + E_{q2})}{2\mathcal{E}_\ell^2} \quad (173)$$

These two quantities are readily measurable by using a 50% beamsplitter that mixes the two single-mode beams the entanglement of which one wants to assess and precisely gives at its output the \pm combinations of the two fields. Two balanced homodyne detections are then needed on these two output ports with the appropriate local oscillator phases to measure the required quadrature fluctuations. Let us stress that these quantities, concerning conjugate variables, are not constrained by a Heisenberg inequality and can be simultaneously very small because they are associated with commuting operators:

$$[\hat{E}_{p1} - \hat{E}_{p2}, \hat{E}_{q1} + \hat{E}_{q2}] = 0 \quad (174)$$

A Gaussian mixed state is non-separable or entangled if one of the two following inequalities is fulfilled:

$$S_{Duan} = \frac{1}{2}(G_-^p + G_+^q) < 1 \quad or \quad S_{Mancini} = G_-^p G_+^q < 1 \quad (175)$$

The first criterion is a particular case of the Duan criterion mentioned in the previous section. The second one has been obtained by Mancini et al.¹⁴. The

¹³L.M. Duan, G. Giedke, J.I. Cirac, P. Zoller, Phys. Rev. Letters **84** 2722 (2000).

¹⁴V. Giovannetti, S. Mancini, D. Vitali, P. Tombesi, Phys. Rev A **67** 022320 (2003).

quantities S_{Duan} and $S_{Mancini}$ are called separabilities. One has $S_{Mancini} \leq S_{Duan}$, so that the Mancini criterion concerns more entangled states than the Duan criterion. To be sure that a Gaussian state is entangled, one therefore needs to show that two non-commuting observables of the system are correlated, and that there is a non-classical correlation ($G < 1$) on at least one of these two observables.

5.4 The Einstein-Podolsky-Rosen argument

5.4.1 Preliminary remark

The EPR argument is based on the relation (174), which implies that quantum mechanics does not forbid the existence of a quantum state which is an eigenstate of the two operators $\hat{E}_{p1} - \hat{E}_{p2}$ and $\hat{E}_{q1} + \hat{E}_{q2}$. In such a state there is a perfect correlation between \hat{E}_{p1} and \hat{E}_{p2} , and a perfect anti-correlation between \hat{E}_{q1} and \hat{E}_{q2} . Therefore *in such a state a homodyne detection on the beam 2 enables us to make a perfect QND measurement on the two conjugate quadratures of beam 1.*

5.4.2 The EPR paper

This famous paper¹⁵ does not deal with the field quadratures, but with the position and momentum variables q_1, q_2, p_1, p_2 which obey the same commutation relations. The authors give the wavefunction of a state that is simultaneously eigenvector of $\hat{q}_1 - \hat{q}_2$ and $\hat{p}_1 + \hat{p}_2$. In this state, as noticed in the previous section, one can perfectly know either the position or the momentum of particle 1 by performing measurements on particle 2 which cannot physically affect particle 1. Their conclusion is that there exists an "element of reality" corresponding to these two quantities, and therefore that Quantum Mechanics which states that it is not possible to attribute simultaneously well defined values to these two quantities, is not complete.

Independently of the question of physical reality raised by the authors, the EPR paper pointed out a very important and puzzling aspect of Quantum Mechanics: when two particles (or two fields) have interacted in the past so that they are described by an entangled state, they form a single physical object even if they are very far apart at the moment of the measurement, and any attempt to separate it into two isolated parts leads to paradoxical

¹⁵A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47** 777 (1935).

conclusions. Let us note however that this statement applies only when one is interested in the *correlations between measurements performed on two particles*. If one is interested in observables of particle 1 only, all the physical predictions can be made when one knows a quantity independent of particle 2, namely the reduced density matrix ρ_1 traced over the Hilbert space of particle 2.

5.4.3 The Reid criterion for EPR correlation

Following M. Reid¹⁶, we will say that a two-mode state exhibits EPR correlations, or is EPR-correlated, when it provides a concrete manifestation of the Einstein-Podolsky-Rosen argument. This means that the state allows us to simultaneously perform QND measurements on the two quadratures of the field which lead to an "apparent violation" of the single-mode Heisenberg inequality. More precisely, the system is EPR correlated when one has the following inequality for the conditional variances:

$$\Delta(E_{p1}|E_{p2})\Delta(E_{q1}|E_{q2}) < \mathcal{E}_\ell^2 \quad (176)$$

Of course, this condition does not violate the rules of quantum mechanics, as it concerns conditional variances and not simple variances.

One can show that, whereas no separable states exhibit EPR correlations, there are entangled states which do not violate the inequality (176). Just like in the single mode case, EPR correlations, related to QND measurements, require more quantum resources than just formal requirements about the state, concerning its non-classical or entangled character. For example losses do not cancel the entanglement between two modes, they just reduce it, whereas they quickly destroy the EPR-correlation.

5.4.4 Generation of EPR-correlated beams from squeezed beams

We have already seen that correlations can be simply created by splitting beams by beamsplitters. In order to get not only correlated, but entangled or even EPR-correlated beams, one needs non-classical beams at their input, such as squeezed states. So we now consider the following set-up: two non-correlated squeezed beams, the first on the p quadrature, the second on the

¹⁶M.D. Reid, Phys. Rev. A **40** 913 (1989).

q quadrature, are incident on a 50% beamsplitter. We note:

$$S_1 = \frac{\Delta^2 E_{p1}}{\mathcal{E}_\ell^2} \quad ; \quad S_2 = \frac{\Delta^2 E_{q2}}{\mathcal{E}_\ell^2} \quad ; \quad M_i = \frac{\Delta E_{pi} \Delta E_{qi}}{\mathcal{E}_\ell^2} \quad i = 1, 2 \quad (177)$$

The quantities S_i are the squeezing factors, and the M_i are related to the purity of the reduced density matrices of beams 1 and 2 (relation (116)).

A short calculation shows that if $S_1 S_2 < 1$ then the Mancini criterion is fulfilled, and the two beams are entangled. This means that, in order to obtain entangled beams, it is enough to have one squeezed state at the input: one can generate an entangled state by simply splitting a single squeezed state on a beamsplitter.

The same calculation also shows that if

$$S_1 S_2 < M_1 M_2 - \sqrt{M_1^2 M_2^2 - 1} \quad (178)$$

then the EPR criterion is fulfilled and the two beams are EPR-correlated. When the two input squeezed states are minimal ($M_1 = M_2 = 1$, the second criterion is identical to the first one: the two beams are EPR-correlated as soon as they are entangled. If they are not pure, more squeezing is needed to counteract the effect of excess noise.

5.4.5 Bell inequalities in continuous variable quantum optics

The EPR paper stated: "While we have thus shown that the wave-function does not provide a complete description of the physical reality, we left open the question of whether or not such a description is possible". J. Bell showed that, if one indeed completes the quantum description by using fluctuating "hidden variables" that account for the stochastic character of the measurements and are independently attached to the systems 1 and 2 when they are far apart ("local hidden variables"), then the following inequality concerning correlations ("Bell inequality") must be fulfilled:

$$S = | \langle \hat{A}_1 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_2 \rangle + \langle \hat{A}_2 \hat{B}_1 \rangle - \langle \hat{A}_2 \hat{B}_2 \rangle | \leq 2 \quad (179)$$

where \hat{A}_i and \hat{B}_i are two different observables of the systems 1 and 2 that have only two eigenvalues ± 1 . Bell showed that in a singlet state of two spins 1/2, S could take the value $2\sqrt{2}$ when \hat{A}_i and \hat{B}_i are the projection of the angular momentum on two appropriately chosen directions. Different

experiments performed on the polarization correlations in entangled twin-photon states exhibited stronger and stronger violations of the Bell inequality (179). These experiments show that a description of reality in terms of hidden local variables is not possible¹⁷.

As stated by Bell himself, the situations is more complex for the position and momentum measurements, or therefore for quadrature measurements, which correspond to operators having a continuous spectrum. Entangled Gaussian states, for example, will not violate Bell inequalities, because they are totally described by a positive Wigner function, for which the quasi-probability is a real probability. The hidden variables exist in such a case: they are the instantaneous value of the quadrature fluctuations, which are "carried" by the beams when they propagate, and therefore are local variables. The situation will be different in non-Gaussian states which cannot be described by a positive Wigner distribution.

The problem remains to find observables with a discrete spectrum ± 1 : the parity of the photon number $(-1)^{\hat{a}^\dagger \hat{a}}$ is a possibility; the sign of the fluctuation of a quadrature component $sign(\delta \hat{E}_q)$ is another one. Both lead to Bell inequalities which have not been so far violated experimentally.

¹⁷A.Aspect, Nature **398** 189 (1999) and refs in

6 MULTIMODE SYSTEMS

6.1 Introduction

We are now interested in a physical system made of N modes, where N can be very large, and which cannot be split in two physically separated parts. On the contrary, all the modes play similar roles, so that the system can be described on the initial mode basis as well as on any other mode basis made of linear combinations of the initial modes. In these systems, we have an interesting new degree of freedom, namely the choice of mode basis. Finding the most appropriate basis, likely to simplify the description of a given physical situation, is in general not obvious. For example, looking at relations (48), we see that the state which is written as a complicated and not factorizable way $|2, 0\rangle/2 + |1, 1\rangle/\sqrt{2} + |0, 2\rangle/2$ on the travelling plane wave mode basis is the simple factorized state $|2, 0\rangle$ on the standing plane wave basis. Note that in the present context the fact the quantum state is entangled is not intrinsic and depends on the mode basis. In the text below, we will call "intrinsic" the properties which do not depend on a special choice of mode basis.

An important example is the quantum description of *optical images*, i.e. of the distribution of the field of a given polarization, in a given plane perpendicular to the propagation axis Oz at $z = z_0$ and depending on the two transverse coordinates (x, y) . One can write the complex scalar field operator in this plane, $\hat{E}^{(+)}(x, y)$, on the TPW basis restricted to the xOy plane:

$$\hat{E}^{(+)}(x, y) = i \sum_{\ell} \sqrt{\frac{\hbar\omega_{\ell}}{2\varepsilon_0 L^3}} e^{i\mathbf{q}_{\ell}\cdot\mathbf{r}} \hat{a}_{\ell} \quad (180)$$

where \mathbf{q}_{ℓ} is the projection of the wave vector on the image plane of components $n_x 2\pi/L$, $n_y 2\pi/L$. It can be also expanded on the Gaussian mode basis:

$$\hat{E}^{(+)}(x, y) = i \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0 L}} v_{p,q}(x, y, z_0) \hat{b}_n \quad (181)$$

where $v_{p,q}(x, y, z_0)$ is the field distribution in a Hermite-Gauss or Laguerre-Gauss mode and the collective index n includes the indices p and q . It can be generally speaking written as:

$$\hat{E}^{(+)}(x, y) = i \sum_m \sqrt{\frac{\hbar\omega_m}{2\varepsilon_0 L}} u_m(x, y) \hat{a}_m \quad (182)$$

where the functions u_m form an orthonormal and complete basis of the transverse plane.

There are other examples of multimode problems in quantum optics: for example the quantum description of light pulses, or of trains of pulses, having an arbitrary temporal shape, which are actually described as superpositions of modes of different frequencies.

6.2 Intrinsic single-mode state

6.2.1 Preliminary remarks

We have already introduced in section (4.1.1) the definition of a single-mode state, which can be written as:

$$|u_1 : \Psi\rangle \otimes |0, 0\dots\rangle \quad (183)$$

It expands over the number states of $n \neq 0$ only in a single mode, that we have labelled u_1 for the sake of simplicity. As seen in the introduction on the simple example of state which is single-mode in the TPW basis and bimodal in the SPW basis, this property is not intrinsic.

Let us consider for example the following multimode single photon state, eigenvector of the total photon number with eigenvalue 1, written as:

$$|\Psi_1\rangle = \sum_m c_m |u_m : 1\rangle \quad (184)$$

with $\sum_m |c_m|^2 = 1$. $|u_m : 1\rangle$ is the state with one photon in mode u_m and zero in all the other modes. Let us introduce the $v_1(x, y)$ function:

$$v_1(x, y) = \sum_m c_m u_m(x, y) \quad (185)$$

It is a first element of a mode basis (v_n), that one completes with orthogonal functions. It is easy to show that in this new mode basis, the previous "multimode" single photon state is:

$$|\Psi_1\rangle = |v_1 : 1\rangle \otimes |0, 0\dots\rangle \quad (186)$$

The same state is in this new basis a regular single mode one photon Fock state.

Let us also consider a multimode coherent state, tensor product of coherent states in all the modes of a given mode basis:

$$|\Psi_2\rangle = |u_1 : \alpha_1\rangle \otimes \dots \otimes |u_m : \alpha_m\rangle \otimes \dots \quad (187)$$

Let us introduce the $w_1(x, y)$ function:

$$w_1(x, y) = \frac{1}{\beta} \sum_m \alpha_m u_m(x, y) \quad (188)$$

with $\beta^2 = \sum_m |\alpha_m|^2$. It is also the first element of a new mode basis (w_n), that one completes again. It is here also easy to show that in this new mode basis, the previous "multimode" coherent state is:

$$|\Psi_2\rangle = |\beta : w_1\rangle \otimes |0, 0\dots\rangle \quad (189)$$

The same state looks now as a single mode coherent state.

The question arises whether all quantum states of light can be reduced to a single mode state by an appropriate choice of the mode basis.

6.2.2 Intrinsic single mode state

a-Definition

An intrinsic single mode state is a quantum state for which there exists a mode basis in which the state vector is of the form $|\phi : v_1\rangle \otimes |0, 0\dots\rangle$ if it is a pure state, or in which the density matrix acts only on the restriction of the Hilbert state to the sub-space of mode v_1 if it is a mixed state. Consequently, the mean number of photons is zero in all the modes except in v_1 .

b-Properties

Let us take a test mode basis (u_ℓ), with the corresponding annihilation operators \hat{a}_ℓ . One can show¹⁸ that a quantum state described by the vector $|\Psi\rangle$ or the density matrix ρ is single mode if and only if the vector space generated by all the vectors $\hat{a}_\ell|\Psi\rangle$ is of dimension 1. For example the state $|1, 1\rangle$ is not single mode, because $\hat{a}_1|1, 1\rangle = |0, 1\rangle$ and $\hat{a}_2|1, 1\rangle = |1, 0\rangle$ are orthogonal.

¹⁸N. Treps, V. Delaubert, A. Maître, J.M. Courty, C. Fabre " Quantum Noise in Multipixel Image Processing " Phys. Rev **A71** 013820 (2005)

A mixed state which is intrinsically single-mode is characterized by the property that all the matrices $\hat{a}_\ell\rho$ are proportional, which implies that there exists a vector $|\phi\rangle$ such that:

$$\forall(\mathbf{r}, t) \quad \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)\rho = \langle \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) \rangle |\phi\rangle \quad (190)$$

Intrinsic single mode states, pure or mixed, have also the following characteristic property of factorization, valid for all couples of modes (k, l) :

$$\langle \hat{a}_k^\dagger \hat{a}_l \rangle = A_k^* A_l \quad (191)$$

which implies the same factorization property for the correlation between the electric field at two different space-time locations:

$$\forall(\mathbf{r}, t, \mathbf{r}', t', i, j) \quad \langle \hat{E}_i^{(-)}(\mathbf{r}, t) \hat{E}_j^{(+)}(\mathbf{r}', t') \rangle = E_i^*(\mathbf{r}, t) E_j(\mathbf{r}', t') \quad (192)$$

i, j being two Cartesian coordinates (x, y, z) of the electric field vector operator.

The physical meaning of an intrinsic single mode state is simple in the case of images: in an intrinsic single mode state, the mean value $\langle \hat{E}^{(+)}(x, y) \rangle$, the standard deviation $\Delta E^{(+)}(x, y)$ and all the higher moments $\left[\langle (\delta \hat{E}^{(+)}(x, y))^n \rangle \right]^{1/n}$ have the same spatial dependence, because they are all proportional to $v_1(x, y)$. In contrast, in non-intrinsic single mode states, these different quantities have different spatial variations.

To experimentally check whether a given quantum state is intrinsically single mode or not, the latter property is more easy to use than the previously mentioned ones: one measures the mean value and the quadrature noise in different areas of the image plane (using a moving blade or an iris in front of the detector for example). If the ratio of these two quantities is not constant when one varies the detection area, then one is sure that the state is not intrinsically single mode. But this test gives only a sufficient condition, and a constant value of the ratio does not imply that the state is single mode.

c-Relation with classical coherence

In classical optics, the notion of coherence¹⁹ is linked to the ability to observe interference fringes with a high visibility, which is contained in the first order correlation function:

$$g^{(1)} = \frac{\langle E^{(+)}(\mathbf{r}, t) E^{(+)}(\mathbf{r}', t') \rangle}{\Delta E^{(+)}(\mathbf{r}, t) \Delta E^{(+)}(\mathbf{r}', t')} \quad (193)$$

¹⁹See Goodman handbook[2] or R. Glauber Les Houches lectures[14]

where the mean $\langle \rangle$ is taken over the classical ensemble of amplitude and phase fluctuations. If $|g^{(1)}| = 1$ then interferences are of contrast 1, and the classical field is said to be perfectly coherent.

The $g^{(1)}$ function can also be defined for quantum fields, replacing the classical mean $\langle E... \rangle$ by the quantum mean $Tr\rho\hat{E}...$. The property of contrast one for the fringes can then be shown²⁰ to be equivalent to the factorization property (192), so that:

$|g^{(1)}| = 1$ if and only if the quantum state, pure or mixed, is an intrinsic single mode state.

Consequently, in order to obtain perfect optical coherence, one does not need a specified quantum state of light, for example a Glauber coherent state. The condition concerns only the intrinsic number of modes characterizing the state. For example, one will be able to see perfect interference fringes with intrinsic single mode states such as a product of coherent states, which is not unexpected, but also with any single photon state, as noticed by Glauber[14]. In contrast, this will not be possible with the state $|1, 1\rangle$. As shown experimentally by Hong, Ou and Mandel²¹, this state produces perfect fringes if one measures them on intensity correlations, sensitive to the $g^{(2)}$ function, and not in a usual interferometer.

6.2.3 Intrinsic number of modes

By extension, we will call *intrinsic number of modes* the dimension of the vector space generated by all the vectors $\hat{a}_\ell|\Psi\rangle$, or all the matrices $\hat{a}_\ell\rho$. An intrinsic two-mode state vector, like $|1, 1\rangle$ for example, can be written, in the appropriate basis, as $|\phi_{12} : v_1, v_2\rangle \otimes |0, 0\dots\rangle$, where $|\phi_{12} : v_1, v_2\rangle$ denotes a two-mode state of the quantum field.

6.3 "Distributed Measurement" in a multimode system

6.3.1 Position of the problem

In a single-mode system, classically characterized by a single complex number, the only quantities that can be measured are "global": they are the total beam intensity or quadrature values, that are measured by a photodetector

²⁰See R.J. Glauber, Les Houches lectures[14].

²¹C. Hong, Z. Ou, and L. Mandel, Phys. Rev. Lett. **59** 2044 (1987).

of large area, covering the entire surface where the mode under consideration has a non-zero amplitude.

A multimode system such as an image is classically characterized by a great number of complex numbers, and there is a great variety of possible measurements: one can measure first local quantities, such as the intensity or quadrature in a small region of the image plane. One can also perform a "distributed measurement" of a physical parameter that modifies the image at different points. More precisely a distributed measurement is made by determining a given function of the different local measurements. This constitutes the very active domain of image processing, with its numerous applications. To make a distributed measurement, one needs a pixellized detector such as a CCD camera, interfaced to a computer or to an analog electronic device which process the data recorded on the photodetector.

6.3.2 Quantum noise in image processing

Quantum fluctuations of light will give rise to fluctuations in the measured quantity that can be readily calculated. Let us take the example of a linear image processing. The measured quantity on the computer output is associated with the operator \hat{M} given by:

$$\hat{M} = \sum_i g_i \hat{N}(x_i, y_i) \quad (194)$$

where $\hat{N}(x_i, y_i)$ is the number of photons measured during the exposure time on the i^{th} pixel centered on point (x_i, y_i) and g_i a positive or negative weighting factor depending on the parameter that one wants to measure. For example it is a widely used technique to take a four-pixel, or quadrant detector, to position the center of a beam of light of any shape. The signal \hat{M}_y with g values $g_1 = g_2 = 1, g_3 = g_4 = -1$ will yield a signal sensitive to the position of the beam in the Oy direction, whereas the values $g_1 = g_3 = 1, g_2 = g_4 = -1$ will yield a signal \hat{M}_x sensitive to the position of the beam in the orthogonal Ox direction. Another example is the determination of the spatial Fourier component of wave vector (k_x, k_y) in the image, using a CCD camera with millions of pixels and the value $g_i = \cos(k_x x_i + k_y y_i)$ for the g_i coefficients.

Even though measurements in multimode systems may look very complicated, the origin of their quantum fluctuations is very simple. It has been

shown²² that *the quantum fluctuations on \hat{M} are due to the fluctuations of a single mode of the field $v_N(x, y)$, called the noise mode of measurement.*

In the case of a measurement of the form (194), the noise mode $v_N(x, y)$ is given by:

$$v_N(x, y) = A \sum_i g_i \langle \hat{E}^{(+)}(x_i, y_i) \rangle \quad (195)$$

where the mean field $\langle \hat{E}^{(+)}(x_i, y_i) \rangle$ is taken in the quantum state of the image, and A is a normalizing factor so that $\int dx dy |v_N(x, y)|^2 = 1$.

In the example of beam positioning of a beam using a quadrant photodetector that we have mentioned above, the noise mode on the signal M_x is the normalized amplitude of the complex electric field of the beam itself in the half plane $x > 0$, and the opposite of the complex electric field amplitude in the half plane $x < 0$. This odd-looking "flipped mode" can be easily generated by inserting a $\lambda/2$ phase plate on half of the initial beam.

In order to reduce the quantum fluctuations on a given measurement below the standard quantum noise limit, it is thus not necessary to squeeze the intensity noise on all the pixels of the detector, which would require enormous quantum resources in the case of a Mega-pixel detector. One just needs to generate a single squeezed state in the noise mode v_N associated with this measurement and to superpose it on the initial image. This scheme has been already successfully implemented in the case of beam positioning using a quadrant detector²³ and has allowed us to position the center of a TEM_{00} laser beam within a few Angstroms.

6.3.3 Optimum measurement of a given parameter

Suppose that we have an image that depends in a distributed way on a given parameter p . For example, a fluorescent nano-object, much smaller than the light wavelength, positioned at the point (x, y) of some object plane, will give in the image plane a broad diffraction pattern, the exact shape of which allows us to know the position of this object with a very high accuracy. Finding the exact image processing scheme which yields the best possible accuracy of the position $p = (x, y)$ of this object from the values of the intensities at each

²²N. Treps, V. Delaubert, A. Maître, J.M. Courty, C. Fabre *Quantum Noise in Multipixel Image Processing* Phys. Rev A **71** 013820 (2005).

²³N. Treps, N. Grosse, C. Fabre, H. Bachor, P.K. Lam, *The Quantum Laser Pointer*, Science **301**, **940** (2003).

point of the image plane and in presence of quantum noise is an interesting optimization problem in image processing at the quantum limit²⁴.

Let us first consider the case where the image is in a coherent state, i.e. when the noise on each pixel is Poissonian (shot noise) and the different pixels have uncorrelated quantum fluctuations. Using well-known techniques of signal theory, one can show that an optimized measurement of p is obtained by a linear image processing protocol of the form (194) with the following value of the gain coefficients g_i :

$$g_i = \frac{\partial}{\partial p} \ln \langle \hat{E}^{(+)}(x_i, y_i, p) \rangle \quad (196)$$

The minimum measurable value of p in presence of shot noise is:

$$p_{min} = \frac{p_0}{2\sqrt{N}} \quad ; \quad \frac{1}{p_0^2} = \int dx dy \left(\frac{\partial}{\partial p} \langle \hat{E}^{(+)}(x_i, y_i, p) \rangle \right)^2 \quad (197)$$

One can show that this value coincides with the Cramer-Rao bound in presence of shot noise. As the Cramer-Rao bound[20] gives a lower limit independently of the precise information protocol used to extract the information, we are sure that there exist no procedure that will be able to give a better signal-to-noise ratio for the measurement of p in the image. But this does not mean that there are no other procedure allowing us to reach the same lower limit. It is actually possible to find a homodyne technique, using a local oscillator of appropriate shape, that also reaches the limit p_{min} given by (197).

In the example of beam positioning, the quadrant detector technique that we have briefly described above using gains ± 1 on the four quadrants gives a standard quantum limit which is 22% above the Cramer Rao bound for the positioning of a TEM_{00} laser beam. It is therefore not the optimum measurement protocol. The Cramer Rao bound is reached when one uses the optimum gain given by (196), which is $g_i = x_i$ for a TEM_{00} beam. The physical reason of this improvement is that the central part of a TEM_{00} beam is not useful if one wants to measure a small displacement of the beam, because it is flat. However, this part of the beam contributes to the noise of the measurement. The weighting factor $g_i = x_i$ of the optimum measurement

²⁴V. Delaubert, N. Treps, C. Fabre, H.A. Bachor, P. Réfrégier *Quantum limits in image processing* Europhysics Letters **81** 44001 (2008).

removes indeed the contribution of this central part, and improves therefore the signal to noise ratio of the measurement.

Finally, one can improve further the measurement of the parameter p by squeezing the noise mode (195), which turns out to be in the case of optimum beam positioning the Hermite-Gauss mode TEM_{10} .

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